CURRENT ELECTRICITY

For Advanced Students

J. C. Mukherjee

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FOR

ADVANCED STUDENTS

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SCIENTIFIC BOOK AGENCY
22, RAJA WOODMUNT STREET
CALCUTTA-700001

Published in 1927 by J. Sinha on behalf of Scientific Book Agency, 22 Raja Woodmunt Street, Calcutta-700001.

Printed at Rajdhani Printing, 117/1, Bepin Behary Ganguly Street, Calcutta-700012.

Price : Rs. 22·50 only

PREFACE

This book is intended to serve the purpose of a suitable and handy text book for Physics Honours course in Current Electricity of Indian universities. It aims to remove the long-felt want of a text book in the true sense for the particular syllabus. The available and often recommended foreign books in this respect contain much more than what is necessary and better serve the purpose of book of reference. Books written by Indian authors (there are very few of them) are unnecessarily voluminous dealing the elementary and advanced topics with similar details and same weightage. This treatise has been so planned as to be different from both of the two types.

The book contains threadbare discussions of the advanced topics with such introduction and in such neat and precise manner that the students joining the degree honours class directly from the pre-university level may find little difficulty in following the course of study and would gather a thorough knowledge and clear conception of the subject.

Alternative treatments including vectorial methods have been incorporated where necessary. Quite a pretty number of numerical examples illustrating important principles and applications thereof have been included as worked-out problems and exercises for students. In representing the vector quantities the convention of using bold types have been followed.

The book has been developed from class-notes prepared and used for a period extending over three decades. This treatise is somewhat a compilation of treatments and methods assimilated by the author during these years of teaching. I would in general way acknowledge my indebtedness to the authors of the works which I freely consulted and relied upon in preparation of the class-notes. A list of such books is appended hereto. This would as well serve the students as a guide to further study.

The book has been hurried through the press amidst numerous present-day handicaps and difficulties. The author had not the scope of reading the proofs in different stages and with as much time and care as is necessary. As such there are some printing mistakes and drawing inaccuracies. For these I solicit the indulgence of the readers. Further, in this first edition of the book omission in treatments and mistakes in calculation of numerical examples may have crept into. The author will remain grateful for any information regarding such as may be detected by the readers. Suggestions and opinions regarding the book is welcome.

I am thankful to the publishers Messrs Scientific Book Agency for their kindly undertaking the publication of the book and to Sri P. L. Sinha of the firm for his initiative in the matter.

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CHAPTER I

D. C. CIRCUITS

I-1. ELECTRIC CURRENT

Drift of charge : Moving charges constitute electric current. The necessary conditions for continuous movement of charge are that (a) the medium should contain free charges capable of movement and (b) there should be an electric field to drive the charges in one particular direction. In solids, generally in metals, there are a few electrons per atom free to move in the interatomic spaces. An electric field inside such a substance would cause a drift of electrons. In liquids, there are no such free electrons and for this reason pure liquids in general (excepting mercury, which is a metal) do not conduct electricity. When some chemical compounds are dissolved in a liquid, the molecules of the solute break up into two parts, carrying equal but mutually opposite charges. These are called *ions*. As for example, sodium chloride (NaCl) when dissolved in water is broken up into positive Na^+ ions and negative Cl^- ions. An electric field inside the solution drives these in opposite directions causing electric current. In gases, there are no free electrons and so gases are ordinarily non-conductors. But, by the action of strong electric fields electrons may be ejected out of atoms by a mechanism involving mutual collisions to form ions. It is only in such a case that a gas conducts electricity.

In order to cause a current to flow through a conductor, an electric field must be maintained inside it. The moving charges gain energy in the field but by collisions with other atoms lose energy and cause generation of heat. For this the source that maintains the field must supply energy for continuous drift of electrons. Such a source maintaining an electric field and supplying energy is called a seat of ***electromotive force.*** This may be obtained by chemical means as in a voltaic cell or

by mechanical process operated in a magnetic field as in an electromagnetic generator. The *emf* of a source creates a potential variation inside the seat of *emf* by transference of charge. In a simple cell the copper plate dipped in dilute sulphuric acid assumes a potential higher than or positive with respect to the acid solution and conversely in a similar way the zinc plate has a potential lower than *i.e.* negative with respect to the solution. The electromotive force in fact causes conversion of non-electrical energy into electrical form and its magnitude is measured as work per unit charge. In transference of charge the electromotive force does work and the charge loses the energy thus acquired when it falls through a potential difference. The electromotive force converts some other forms of energy into electrical energy and potential difference causes conversion of electrical energy into other forms of energy. The electromotive force is the cause and potential difference is its effect. Both are measured by the same unit and the practical unit of such measurement is calculated as *one joule per coulomb* which is called a **Volt.**

Process of electrical conduction : It is instructive to consider the mechanism of flow of electric current in metals. The free electrons in absence of any electric field move at random in all possible directions. If the metal is subjected to an external field the electrons are urged to move in one direction. Thus there is a drift of electric charge. This constitutes electric current. The conventional electric current is taken to flow in the direction opposite to the drift of electrons. For continual encounters of the electrons with the atoms of the metal the momentum of moving electrons does not increase indefinitely.

Let m the mass of an electron moving forward under the electric field with an average velocity u. If n be the number of free electrons per unit length of the conductor, the total momentum of all moving charges in unit length is nmu. Let e be the charge of an electron. The total force acting on n electrons in an electric field of intensity E is Ene, and this is the measure of the rate at which the total momentum of all the electrons in unit length is increased. Again, the collisions

cause a dimunition in momentum and this may be taken to be a factor of number of electrons and their velocity and so proportional to nu. If β represent the constant of this proportionality depending upon the interval between successive collisions, then the rate of decrease of momentum is βnu. So the equation representing the rate of increase of momentum may be expressed as

$$\frac{d}{dt}(\,nmu\,) = Ene - \beta nu.$$

If i the current strength be defined as the charge flowing through any fixed section of conductor in unit time, then

$$i = nue$$

$$\text{or} \quad nu = \frac{i}{e}$$

$$\text{or} \quad nmu = \frac{im}{e}$$

$$\text{or} \quad \frac{d}{dt}(\,nmu\,) = \frac{m}{e} \cdot \frac{di}{dt}$$

$$\text{or} \quad \frac{di}{dt} = \frac{e}{m} \cdot \frac{d}{dt}(\,nmu\,)$$

$$\text{or} \quad \frac{di}{dt} = \frac{e}{m}(Ene - \beta nu) = \frac{e}{m}\left(E\,ne - \frac{\beta i}{e}\right)$$

$$\text{or} \quad \frac{di}{dt} = \frac{ne^2}{m}\left(E - \frac{\beta}{ne^2}\,i\right)$$

$\frac{di}{dt}$ becomes zero, if i attains the value of $\frac{ne^2E}{\beta}$. This condition states that the velocity of the drift of electrons attains a steady value when the rate of flow of charge reaches a value $i = \frac{ne^2E}{\beta}$. This condition may be written as $E = \frac{\beta i}{ne^2}$.

If the rate of fall of potential (V) along the conductor is taken into account, electric intensity is expressed as

$$E = -\frac{dv}{dx} = \frac{\beta}{ne^2}i$$

$$\text{or} \quad dv = -\frac{\beta}{ne^2}\,i.dx$$

Integrating between two points separated by a finite length l inside the conductor, we get for the potential difference (V) between the points, the expression

$$V = \frac{\beta l}{ne^2} i = Ri, \quad \text{where } R = \frac{\beta l}{ne^2}$$

This equation shows that the current though caused by a potential difference, its strength is controlled by a factor R, depending upon the length of the conductor, the frequency of collisions, the number of free electrons per unit length and the electronic charge. This constant of proportionality R has been called **Resistance** of the conductor.

Equation of Continuity : If the current through unit area of a conductor, called *current density* is denoted by I, the total current through an area dS may be expressed as

$$i = \iint \mathbf{I}.\, d\mathbf{S}$$

If dS be an area enclosing an arbitrary volume dv, the current flowing through the surface is given by the rate of decrease of total charge in the volume dv, so

$$i = \iint \mathbf{I}.d\mathbf{S} = -\frac{\partial}{\partial t} \iiint \sigma.dv,$$

σ being the volume density of charge inside the element dv. But by Gauss' theorem, we may write

$$\iint \mathbf{I}.d\mathbf{S} = \iiint div.\ \mathbf{I}.\ dv$$

Hence $$\iiint \left[div\ \mathbf{I} + \frac{\partial \sigma}{\partial t} \right] dv = 0$$

or $$div\ \mathbf{I} + \frac{\partial \sigma}{\partial t} = 0$$

This is the *equation of continuity*. If $\frac{\partial \sigma}{\partial t} = 0$, *i.e* if there be no change in volume density of charge, as it occurs in free space, where there is no sink or source of charge, $div\ \mathbf{I} = 0$

1-2. RESISTANCE

Ohm's Law : The current through a conductor though caused by the potential difference at its ends, is controlled by the property of the conductor. This fact was formulated by G. S. Ohm, a German physicist in 1826, in the form of a law stated as follows :

When a steady current flows through a conductor the potential between its ends is directly proportional to the current provided that the physical conditions of the conductor does not change.

Thus if V is the potential difference and i the current, then

$$V \propto i$$

$$\text{or} \quad V = iR$$

R is a constant, depending upon the nature and physical conditions of the conductor, called its *Resistance.*

This expression may be written as,

$$i = \frac{V}{R} \quad \text{or} \quad R = \frac{V}{i}$$

Any of these equations represents the Ohm's law in mathematical form. This law connects the three fundamentals, the cause (*emf*), the control (resistance) and the effect (current) concerning continuous flow of electric charge through a coductor.

Practical units : The current is measured in ampere, *emf* or P. D. in volt and the resistance in Ohm. A conductor has a resistance of one ohm if one ampere current flowing through it causes a potential difference of one volt at its ends.

Each of these units is defined in two ways. The unit obtained from theoretical laws direct are called ***true units*** and those fixed from practical agreed methods of measurement are ***International units.*** These will be discussed hereafter in proper places. The practical standards *i.e.* international units are defined here.

International Ohm is the resistance of a column of mercury at the temperature of melting ice (0°C) 14·4521 gm. in mass

of uniform cross-sectional area comprising a length of 106·300 cm. It is slightly greater than true ohm.

International ampere is that unvarying current which when passed through an aqueous solution of silver nitrate deposits silver at the rate of 0·00111800 gm. in one second. Consequently *international coulomb* is the amount of charge carried by one international ampere current in one second.

International volt is the potential difference which drives a current of one international ampere when applied across a resistance of one international Ohm. It is in practice realised in terms of *emf* at 20°C of standard cadmium cell, taken to be 1·0183 volts.

Resistance and Conductance : The reciprocal of a resistance of a conductor (called resistor when its resistance is being considered) is its conductance. If R ohms is the resistance of a piece of material, its conductance K is expressed as $K=\frac{1}{R}\,ohm^{-1}$. Conductance is also expressed in unit called *mho*. A resistor of one ohm resistance is a conductor of one mho conductance.

Resistivity and Conductivity : If R is the resistance of a resistor of length l and cross-sectional area A, then R is found to vary as l/A. Introducing a constant of proportionality this is expressed as $R=\rho l/A$. The constant term ρ is called the *specific resistance or resistivity* of the material. It may be defined as the resistance of the material of unit length and unit cross-section. In practical units it is the resistance of one centimetre cube of the material concerned and is expressed in ohm-centimere unit. Conductivity is the reciprocal of resistivity.

Factors affecting resistance : Resistivity of a substance is a constant depending upon the material and its physical conditions such as temperature, tension and on environments such as light and magnetic field. Change in crystalline structure has a pronounced effect on resistivity.

EFFECT OF TEMPERATURE : The variation of resistance with temperature is respresented by the equation

$$R_t = R_0(1 + \alpha t + \beta t^2)$$

R_t and R_0 are the resistances at t° and O° measured in celsius (centigrade) scale in a gas (air) thermometer, and α, β are the constants for a particular material. For pure metals β is a small quantity and for alloys like constantan and manganin α also tends to zero. It is for this reason that standard resistance coils are constructed of wires of these alloys. The variation of resistance with temperature though parabolic in nature, within small range it may be taken as linear, expressed by the equation $R_t = R_0\ (1 + \alpha t)$. The effect of temperature on resistance is used for temperature measurement in *platinum thermometer.* Carbon has a *negative* temperature coefficient, *i. e.* for it α is negative and the resistance of carbon decreases with rise of temperature.

Supra-conductivity is the property of metals of losing their resistance suddenly at a very low temperature approaching absolute zero. For mercury this occurs at 4°·22K. This property is shown by lead, tin and several other metals. A current flowing through a circuit made up of a super-conductive material at the appropriate temperature would continue even when the source of *emf* has been removed. This is the effect of lack of resistance.

EFFECT OF LIGHT : Though incident light has no effect on resistance of most metals, selenium is an exception. Resitivity of selenium diminishes according to the intensity of light falling on it.

Selenium cell is a contrivance utilising the property of light used to convert light energy into electrical energy.

EFFECT OF MAGNETIC FIELD : Magnetic field slightly affects the resistance but it is much pronounced in case of bismuth. A strong magnetic field applied at right angles to the direction of current may increase the resistance of a bismuth resistor to double its normal value.

EFFECT OF PRESSURE AND TENSION: Tension increases the resistance but pressure has little effect.

Grouping of Resistors: Two or more resistors may be joined either to inciease or decrease the resultant effect.

RESISTANCES IN SERIES : When several resistors are so arranged that the same current flows through all of them, they are said to be *in series*. In suçh an arragement the separate resistors are joined end to end one after another, so that the free ends of the first and the last of the group respectively form the terminals of the combination. In such a grouping (Fig 1.1a) the total length of the conductor increases and hence there is an increase in the effective resistance, which is obtained as the sum of the individual resistances. It may be proved as shown below.

If i be the current through the resistors, the total drop of potential at the terminals A and B of the resistors is obtained as the sum of the potential drops in the individual resistances R_1, $R_2, R_3 \cdots$ *i.e*, between A–C, C–D, D–B etc. So the relevant potential drop V is given by

$$V = iR_1 + iR_2 + iR_3 + \cdots$$

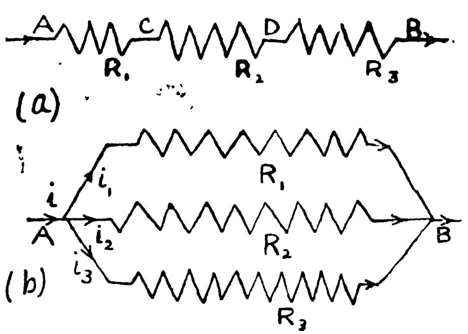


Fig. 1·1

If a single resistance R be substituted for the combination so as to produce the same potential drop at the terminals for the same current, then

$$iR = iR_1 + iR_2 + iR_3 + \cdots$$

$$\text{or} \qquad R = R_1 + R_2 + R_3 + \cdots$$

RESISTANCES IN PARALLEL : When several resistors are so joined that each of them becomes an independent path for a part of the current obtained from the source to which they are connected independent of one another (fig. 1·1b). The

total current flowing through the combination is the sum of the current in the alternative paths. In such a case the effective cross-section of the conductor may be regarded as to be increased and as such the equivalent resistance decreases in relation to any of the individual resistances. Let i_1, i_2, i_3...be the currents in the respective branches having resistances R_1, R_2, R_3...

The total current supplied by the source is given by

$$i=i_1+i_2+i_3+\cdots\cdots$$

If V is the potential drop at the ends of any of the resistors, which is the same for all, then

$$i=\frac{V}{R_1}+\frac{V}{R_2}+\frac{V}{R_3}+\cdots\cdots$$

Let R be a single resistance such that when joined independently between two points having the same potential difference *i.e.* with the terminals of the same source of *emf*, it produces the same current, then,

$$i=\frac{V}{R}=\frac{V}{R_1}+\frac{V}{R_2}+\frac{V}{R_3}+\cdots\cdots$$

Hence, $$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\cdots\cdots$$

Remembering that conductance (K) is the reciprocal of resistance, the above relation may be expressed as

$$K=K_1+K_2+K_3+\cdots\cdots$$

If there be two resistors of magnitudes R_1 and R_2 joined in parallel, then the equivalent resistance is given by

$$R=\frac{R_1R_2}{R_1+R_2}$$

It may be helpful to remember that the equivalent resistance of several resistances in parallel is less than the resistance of any of the components.

1-3. CURRENT DISTRIBUTION IN NETWORK CIRCUIT

Kirchhoff's Laws : Problems concerning the manner in which current flowing in a network consisting of junctions distributes itself in different branches may be solved by two laws enunciated by Kirchhoff.

FIRST LAW : *In any network of conductors the sum of the currents which meet any point is zero.*

Let O be the junction of four conductors (fig. 1·2a), the currents in the respective branches being i_2, i_3, i_4 Considering the current flowing into a junction as positive and taking it

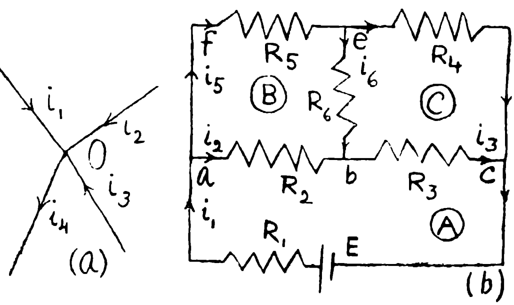


Fig. 1·2

as negative when it flows away from it, then as shown in the circuit diagram according to the law stated in the above, we may write

$$i_1+i_2+i_3-i_4=0$$

or in short, $\Sigma i \;=0$

Since current indicates steady flow of charge there cannot be any accumulation of charge at any junction, the first law refers to this fact.

SECOND LAW : *The algebraic sum of the electromotive forces in any closed circuit or mesh is equal to the algebraic sum of the products of resistance of each portion of the circuit and current flowing through it.*

A network circuit is shown in diagram (fig. 1·2b). Applying the second law respectively to the closed meshes indicated as A, B, C in the diagram, we get,

$$i_1R_1+i_2R_2+i_3R_3=E \qquad \ldots \qquad \ldots \qquad \text{(a)}$$

$$i_5R_5+i_6R_6-i_2R_2=0 \qquad \ldots \qquad \ldots \qquad \text{(b)}$$

$$i_4R_4-i_3R_3-i_6R_6=0 \qquad \ldots \qquad \ldots \qquad \text{(c)}$$

The second law may be briefly expressed in the form $\Sigma\,(E-ir)=0$. Kirchhoff's second law follows from the principle

of conservation of energy. It would be clear if we remember that *emf* in a circuit supplies energy to moving charge and on the contrary a moving charge dissipates energy in falling though a potential difference. The second law implies that the energy supplied by the source of *emf* is equal to the energy dissipated by the current, since the law may be expressed as

$$\Sigma\,(\,Ei - i^2r\,) = 0.$$

Maxwell's modification : To simplify the problem of current distribution in a complicated circuit Maxwell introduced the idea of 'cyclic current'. According to this,

In each mesh a cyclic current of some specific value is imagined to flow, all being clockwise and regarded as positive. The emf encountered by the traversing current is reckoned negative, if it tends to send a current in the direction opposite to flow.

The principle is illustrated in the circuit shown in fig 1·3. These are two meshes marked as A and B. The currents in these are denoted respectively by x and y. Through the resistance R_2 the cyclic currents are in opposition mutually.

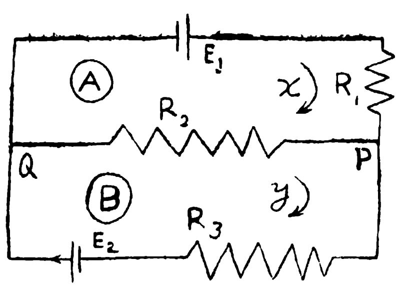


Fig. 1·3

In the mesh A : $R_1x + R_2\,(x - y) = E_1$

In the mesh B : $R_3y + R_2\,(y - x) = E_2$

Numerical example : Consider the case in which $R_1 = 2$ ohms, $R_2 = 6$ ohms, $R_3 = 5$ ohms and $E_1 = 2$ volts, $E_2 = 1$ volt. The current through R_2 and the potential difference at its ends are obtained by solving for x and y.

$$2x + 6(x - y) = 2, \quad \text{or} \quad 8x - 6y = 2$$

$$5y + 6(y - x) = 1, \quad \text{or} \quad 6x - 11y = 1$$

Solving we get $x = \frac{4}{13}$ amp. and $y = \frac{1}{13}$ amp. Hence current through PQ flows from P to Q and is of magnitude $\frac{3}{13}$ amp.

Potential difference between P-Q is $\frac{6 \times 3}{13}$ volts $= \frac{18}{13}$ volts.

Illustrative Examples

1. *Two batteries of emf 3 and 4 volts having internal resistances 2 ohm and 1 ohm respectively are joined in opposition by two wires of resistances 12 and 6 ohms. If the electric mid-points of these two wires are joined by a conductor of 10 ohms resistance, find the current in this wire and the potential difference at its ends.*

Solution : The currents in different branches are as shown in the circuit diagram (Fig 1·4). Applying Kirchhoff's second law in the meshes APQB and CQPD respectively we get,

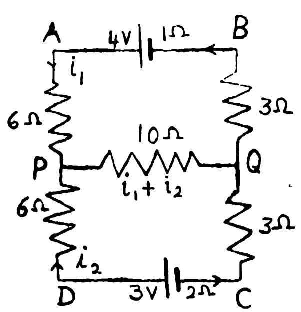


Fig. 1·4

$$6i_1+10(i_1+i_2)+3i_1+i_1=4$$
$$6i_2+10(i_1+i_2)+3i_2+2i_2=3$$

Rearranging, $20i_1+10i_2=4$
$$10i_1+21i_2=3$$

Hence by solving, $i_1=\frac{27}{160}$ amp., $i_2=\frac{1}{16}$ amp. So the current through the 10-ohm resistor is $i_1+i_2=\frac{27}{160}+\frac{1}{16}=\frac{37}{160}$ amp.

and the p. d. at ends is $\frac{37}{160}\times 10=\frac{37}{16}$ volt.

2. *Twelve wires each having a resistance R are joined to form a cube. Find the effective resistance between (a) two diagonally opposite corners, (b) two opposite corners of the same face when an emf is applied respectively between the points concerned.*

Solution : (a) As shown in the diagram (fig 1·5a) the current at A divides itself in three branches and from symmetry distribution is same in all of them. Hence if i is the current entering at A, the current in each of AF, AB, AD are $i/3$ in magnitude. Similarly the current reaching H by the three branches GH, CH, EH are each of magnitude $i/3$. It is easy to find that the currents in DE, DC, EF, FG, BG are of

magnitude $i/6$ in each of them. Applying Kirchhoff's law between A-H by any path (say ABCH) we get,

$$\frac{1}{3}Ri+\frac{1}{6}Ri+\frac{1}{3}Ri=V$$

$$\text{or}\quad V=\frac{5}{6}Ri$$

If the effective resistance between A-H be r, then

$$ir=V=\frac{5}{6}Ri. \qquad \text{So,}\quad r=\frac{5}{6}R.$$

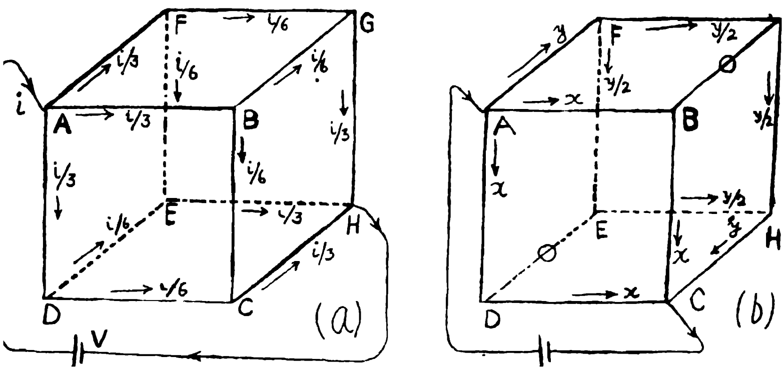


Fig. 1·5

(b) The current entering A is divided in three branches as x, x and y and from symmetry the current leaving C will be due to similar contributions. The currents in different branches are as shown (fig. 1·5b). It will be evident from distribution that DE and BG carry no current and hence we may ignore them. FEH and FGH are circuits each carrying two resistances R in series and these are themselves parallel to each other. Their effective values between the points F-H is R. This again is in series with AF, HC. The eight resistances at the back of ABCD are equivalent to $3R$. Considering the resistances in the front face ABCD, it is found that the resistances between A-C are (AB+BC) in parallel with (AD+DC), each of the combination being equal to $2R$. So these four resistances are equivalent to R. Hence the resistance between A-C may be considered to amount to $3R$ and R in parallel, which are equivalent to $\frac{3}{4}R$.

1-4. SETTING UP OF A BATTERY

Grouping of cells : More than one cells when combined to send a current jointly through a circuit are said to form a *battery of cells*. This grouping may be done in three different ways.

CELLS IN SERIES : If n cells each of internal resistance r and *emf* E are connected in such a way that the internal resistances are all in series (for this purpose positive pole of one is connected to the negative of the next) the cells are said to be in series (fig. 1·6a). If the circuit is completed to obtain a current in an external resistance R, the current i is given by

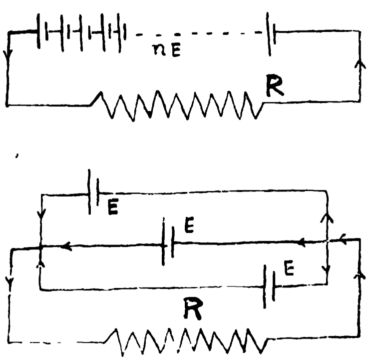


Fig. 1·6

$$i=\frac{nE}{R+nr}$$

When $r\rightarrow 0$, $i=\frac{nE}{R}$, which is n-times the current available from a single cell.

CELLS IN PARALLEL : If n cells each of internal resistance r and *emf* E are joined in such a way so that the internal resistances fall in parallel (for this all the positive poles are joined at one point to form one terminal and the negative poles similarly joined become the other terminal) the cells are said to be in parallel (fig. 1·6b). If the combination sends a current through an external resistance R, the current i is given by

$$i=\frac{E}{R+\frac{r}{n}}=\frac{nE}{nR+r}$$

when $r\rightarrow 0$, $i=\frac{E}{R}$, which is same as the current available from a single cell.

It may be noted that cells in series increase the current

strength and cells in parallel do not supply a stronger current but increase the amper-hour capacity.

MIXED CIRCUIT : Let $(m \times n)$ number of cells beso grouped that m rows of cells each containing n cells in series, are joined in parallel. If r be the internal resistance of any individual cell and R the external resistance through which the battery sends a current i, then

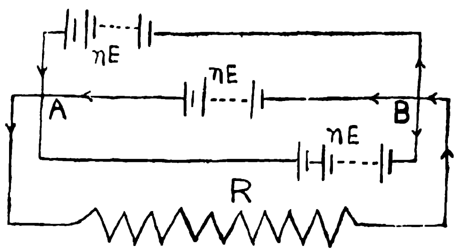


Fig. 1·7

$$i = \frac{nE}{R + \frac{nr}{m}} = \frac{mnE}{mR + nr}$$

$$\text{or} \quad i = \frac{mnE}{[\sqrt{mR} - \sqrt{nr}]^2 + 2\sqrt{mRnr}}$$

For maximum value of i, we should have $\sqrt{mR} = \sqrt{nr}$, that is $R = nr/m$.

Thus for maximum current the external resistance should be equal to the total internal resistance of all the cells.

Lost volts in a battery : If a battery is connected to send a current through an external resistance (called the load), the potential difference across the load is less than the *emf* of the battery as obtained in open circuit, because of the drop of voltage in the internal resistance of the battery. If a battery of *emf* E and internal resistance r delivers a current i through a load of resistance R, the voltage drop (V) across the load is given by

$$V = E - ir \quad \text{and} \quad i = \frac{V}{R}$$

$$\text{So,} \quad V = E - \frac{V}{R} r$$

$$\text{or} \quad V = \frac{E}{1 + r/R}, \quad \text{if } r \to 0, \ V = E.$$

Illustrative Examples

1. *Find the least number of cells each of emf 1·55 volts and internal resistance 0·7 ohm necessary for a glow lamp requiring a current of 2 amperes and a potential difference of 10 volts at its ends.*

Solution : Resistance of the lamp $R=\frac{10}{2}=5$ ohms

$$i=\frac{nE}{R+nr/m}, \text{ for maximum current } R=\frac{nr}{m}$$

$$\text{So, } i=\frac{nE}{2R} \quad \text{or} \quad 2=\frac{1{\cdot}55n}{2\times 5}, \quad \text{hence } n=12 \text{ or } 13$$

$$\text{(i)} \quad m=\frac{nr}{R}=\frac{12\times 0{\cdot}7}{5}=1{\cdot}68$$

$$\text{(ii)} \quad m=\frac{nr}{R}=\frac{13\times 0{\cdot}7}{5}=1{\cdot}82$$

Hence the least possible value of m is 2.

So the number of cells $=m\times n=2\times 12=24$.

2. *A battery of 18 identical cells is formed by grouping 6 cells in series and connecting 3 such groups in parallel. If each cell has an emf of 2 volts and internal resistance 1 ohm, what will be the value of the current through an external resistance of 4 ohms ? What is the potential difference across the battery terminals ?*

Solution : $i=\frac{mnE}{mR+nr}=\frac{18\times 2}{3\times 4+6\times 1}=\frac{36}{18}=2$ amp.

P. D. at the ends of the resistance is $2\times 4=8$ volts.

Emf of the battery is $6\times 2=12$ volts.

Hence p. d. across the battery is $i\times r=2\times 2=4$ volts.

1·5. D. C. BRIDGES

Wheatstone's net : Four resistors P, Q, R, S along with a battery and a galvanoneter are arranged in six branches to form a circuit in a trilateral symmetry about a common point O as shown (fig 1·8). The branches LO and MN respectively accommodate a galvanometer and a source of *emf* (battery).

The resistance of the four other branches LM, LN, MO, NO are respectively P, Q, R, S. The resistance of the battery and of the galvanometer are respectively B and G. The *emf* of the battery is e.

Let the current in the branch MO be denoted by x and those in the battery and the galvanometer circuits be respectively b and g. The direction of the current in the different branches are as shown in the diagram.

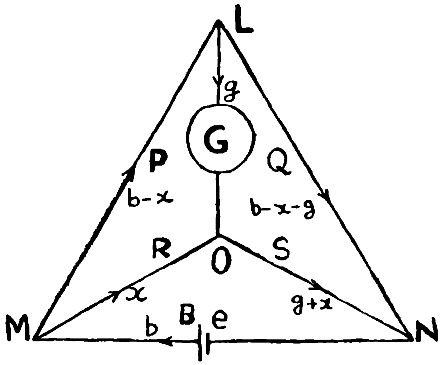


Fig. 1·8

Applying Kirchhoff's first law, we find

Current in ML $= b - x$

Current in LN $= b - x - g$

Current in ON $= g + x$

Applying Kirchhoff's second law in the meshes LMN, LMO and LNO respectively we get,

$$
\begin{aligned}
Bb + P(b-x) + Q(b-x-g) &= e \\
P(b-x) + Gg - Rx &= 0 \\
Q(b-x-g) - S(g+x) - Gg &= 0
\end{aligned}
$$

Rearranging the terms,

$$
\begin{aligned}
(B+P+Q)b - (P+Q)x - Qg - e.1 &= 0 \\
Pb - (P+R)x + Gg + 0.1 &= 0 \\
Qb - (Q+S)x + (G+S+Q)g + 0.1 &= 0
\end{aligned}
$$

Solving for g we get,

$$
g = \frac{\begin{vmatrix} (B+P+Q) & -(P+Q) & -e \\ P & -(P+R) & 0 \\ Q & -(Q+S) & 0 \end{vmatrix}}{\begin{vmatrix} (B+P+Q) & -(P+Q) & -Q \\ P & -(P+R) & +G \\ Q & -(Q+S) & -(G+S+Q) \end{vmatrix}}
$$

Evaluating this determinant form of expression we shall get the value of g, the galvanometer current under any arrangement. The condition for null *i.e.*, no current through the galvanometer may be obtained by equating the numerator of the above expression to zero.

The numerator of the above expression is

$$+e.\begin{vmatrix} P & -(P+R) \\ Q & -(Q+S) \end{vmatrix} = e.[-P(Q+S)+Q(P+R)]$$

or the numerator $=e(QR-PS)$

The numerator vanishes if $QR=PS$

So the galvanometer current $g=0$, if $\dfrac{P}{Q}=\dfrac{R}{S}$

This is the condition for a balance in the bridge. Now if the four resistors are so chosen so as to make the galvanometer current zero, then we know $S=QR/P$. This shows that if three resistances P, Q, R are known, the value of S an unknown quantity may be obtained. This is one of the most commonly used methods for comparison of resistances.

It should be noted that if one pair of resistances (P-Q or R-S) be extremely small, the numerator will be a small quantity and the arrangement will be insensitive under all conditions. This difficulty cannot be removed by increasing the other pair as that would again increase the denominator. Hence Wheatstone bridge is unsuitable for comparison of low resistances. It would be also advantageous to have the battery and galvanometer resistance as low as possible as this would make the denominator small.

Wheatstone bridge is unsuitable for comparison of high resistances as well, for the high resistances put in one pair of arms would cause a very small current to pass through the galvanometer making the arrangement insensitive.

In any Wheatstone bridge arrangement the unknown resistance is placed in the branch containing S, called the fourth arm. The branches comprising P and Q are known as ratio arms and the branch representing R is the third arm.

Sensitivity of the bridge : Sensitivity implies that for a slight deviation from the balanced condition the current flowing through the galvanometer should be appreciable. In other words the ratio $g:s$ (galvanometer current : fourth arm current) should be maximum. This may be obtained either by particular choice of the galvanometer and battery positions or by applying suitable values of the resistances in the ratio arms. The two cases are being discussed separately.

RELATIVE POSITIONS OF BATTERY AND GALVANOMETER : In the expression for g, a smaller value of the denominator would cause relative increase of the value of g. Let in an arrangement the galvanometer be joined between P-Q and R-S junctions and the battery between P-R and Q-S junctions (fig 1·9a). Under this arrangement the denominator of expression for g is obtained as

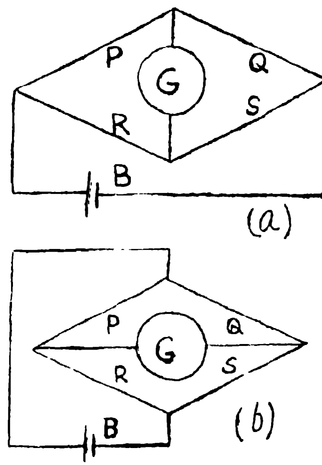


Fig. 1·9

$$D_1 = \begin{vmatrix} P+Q+B & -(P+Q) & -Q \\ P & -(P+R) & +G \\ Q & -(Q+S) & -(G+S+Q) \end{vmatrix}$$

$$\text{Or,} \quad D_1 = (P+Q+B)\,[(P+R)\,(G+S+Q)+G(Q+S)] \\ -(P+Q)\,[P(G+S+Q)+GQ] \\ +Q[P(Q+S)-Q(P+R)]$$

Rearranging the terms,

$$D_1 = BG(P+Q+R+S)+B(P+R)(Q+S)+G(P+Q)(R+S) \\ -[PQ(R+S)+RS\,(P+Q)]$$

If the positions of the battery and the galvanometer be interchanged (as in fig. 1·9b), the new denominator is obtained as

$$D_2 = BG(P+Q+R+S)+G(P+R)(Q+S)+B(P+Q)(R+S) \\ -[PQ(R+S)+RS(P+Q)]$$

Hence $D_2 - D_1 = G(P+R)(Q+S) + B(P+Q)(R+S)$
$$- B(P+R)(Q+S) - G(P+Q)(R+S)$$

By simplification,

$$D_2 - D_1 = (G-B)[(P+R)(Q+S) - (P+Q)(R+S)]$$

Or, $\quad D_2 - D_1 = (G-B)(P-S)(Q-R)$

If $D_2 > D_1$, that is if $D_2 - D_1$ is positive, the first arrangement becomes more sensitive. Hence for better sensitivity the expression $(G-B)(P-S)(Q-R)$ should be positive. This indicates that if $G>B$, P should be greater than S and Q greater than R (or conversely $S>P$ and $R>Q$). So it is to be noted that the first arrangement becomes sensitive if in that arrangement the galvanometer, which has a resistance greater than that of battery, is placed between P-Q and R-S junctions, when $P>S$ and $Q>R$.

Thus for the better sensitivity whichever of the two, battery and galvanometer, has the greater resistance must be placed between the two junctions formed by two greater resistances and two smaller resistances respectively.

Although the method provides for making the galvanometer current maximum in a slightly unbalanced condition, it is found that the arrangement also makes the current through the resistance to be measured appreciably greater. This is particularly noticeable when the battery has a negligible resistance. Thus the arrangement is unsuitable in cases where the heating effect of the current on the resistance to be measured is not a negligible affair as in the case of a platinum resistance thermometer.

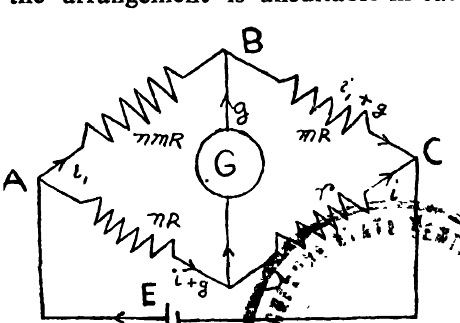


Fig. 1[illegible]

CHOICE OF RATIO-ARMS RESISTANCES : Let R be the resistance to be measured and n the ratio of the ratio-arms resistances. If nR be the resistance

in the third arm in the balanced condition, then $R=r$. In any unbalanced state let $R \sim r = \Delta r$. Let the ratio-arms resistances expressed in terms of R, be written as mR and $n(mR)$. Let i be the current in r, g the current through the galvanometer which has a resistance G. The current in the third arm is $(i+g)$ and the currents in the two ratio-arms are respectively i_1 and (i_1+g) flowing in directions indicated in the circuit diagram (fig. 1·10).

Applying Kirchhoff's law in the meshes ABD and BCD respectively, we get,

$$n(mR)i_1 - Gg - nR(i+g) = 0$$
$$mR(i_1+g) - ri + Gg = 0$$

Eliminating i_1 from these two equations,

$$\frac{g}{i} = \frac{n(r-R)}{(n+1)G + n(1+m)R} = \frac{\Delta r}{(1+\frac{1}{n})G + (1+m)R}$$

The ratio g/i is a measure of sensitivity. For a small defect of balance measured as Δr, this ratio should be maximum for better sensitivity. This indicates that n should be large and m small. This implies that *the ratio of the resistances in the ratio-arms should be large and also that the smaller resistance of these should not be high in comparison with r*. This determines the condition for better sensitivity. We may consider the two limiting cases when (i) the three resistances P, Q, R are of equal magnitude and (ii) the ratio is too great.

(i) If $n=m=1$, $\dfrac{g}{i} = \dfrac{r-R}{2(G+R)}$

(ii) If $n \to \infty$, $m \to 0$, $\dfrac{g}{i} = \dfrac{r-R}{G+R}$

The advantage in the second case is not very great.

BEST VALUE OF GALVANOMETER RESISTANCE : In the expression for sensitivity shown in the above G and R both occur in the denominator. We may consider what value of G should give the best advantage.

Let us consider two galvanometers having coils of same size and mass, one being a single turn coil and the number of turns in the other being p. Such increase in number of turns causes increase in resistance p^2 times. This is due to p times increase in length and $\frac{1}{p}$th dimunition in cross-section. So the galvanometer resistance may be taken to be proportional to p^2 *i.e.* $G \propto p^2$. This may as well be written as $p \propto \sqrt{G}$.

Again, the deflection θ of the coil for a given current is proportional to the number of turns and also to the current strength, that is

$$\theta \propto g.p \quad \text{or,} \quad \theta \propto g\sqrt{G}$$

The expression for the galvanometer current has been shown to be given by

$$g=\frac{i\,(r-R)}{(1+\frac{1}{n})G+(1+m)R}$$

So when i and $(r—R)$ are not changed

$$\theta \propto \frac{\sqrt{G}}{(1+\frac{1}{n})G+(1+m)R}$$

$$\text{or, } \theta = \frac{k\sqrt{G}}{(1+\frac{1}{n})G+(1+m)R}$$

$$\text{Hence } \frac{d\theta}{dG}=\frac{\frac{1}{2}k}{\sqrt{G}[(1+\frac{1}{n}G+(1+m)R]}-\frac{(1+\frac{1}{n})\sqrt{G}.k}{(1+\frac{1}{n})G+(1+m)R]^2}$$

For a maximum deflection $\frac{d\theta}{dG}=0$

$$\text{So, } \frac{1}{2\sqrt{G}}-\frac{(1+\frac{1}{n})\sqrt{G}}{(1+\frac{1}{n})G+(1+m)R}=0$$

$$\text{or } \quad G=\frac{1+m}{1+\frac{1}{n}}.R$$

This shows that G *should be of the same order of magnitude as R*. If n is never less than 1 or m never greater than 1,

the limiting values of G are $2R$ (when $m=1$, $n=\infty$) and $0{\cdot}5R$ (when $m=0$ and $n=1$).

Kelvin's Double bridge : Wheatstone's bridge is not suitable for measurement of low resistances (of the order of 0·1 ohm and less) because of (a) insensitiveness of the bridge and (b)

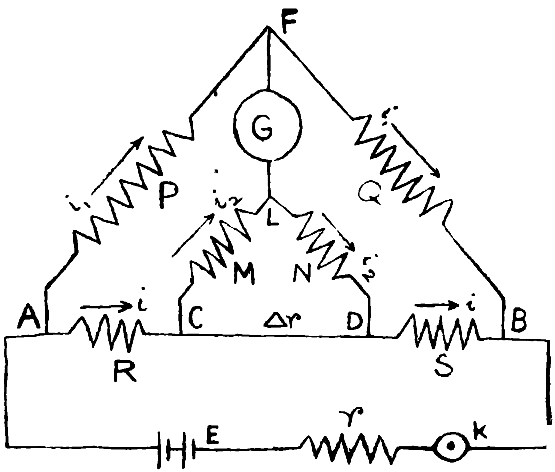


Fig 1·11

errors due to inclusion of resistances of the junctions and connecting wires in the bridge arms. In Kelvin's double bridge these errors are eliminated. The essential parts of the bridge are as shown in the circuit diagram (fig. 1·11). S is an unknown resistance and R is a standard resistance both having extra potential leads. A metal bar CD having small resistance (Δr) joins these two resistors. Four known resistances P, Q, M, N are joined through a low resistance galvanometer to form the bridge.

Let us assume that the bridge is balanced. In such a state the currents as shown in the diagram flow through the several resistances and it is same through R and S. Applying Kirchhoff's law to the meshes ACGF and BDGF,

we get,

$$Pi_1 - Mi_2 - Ri = 0 \quad \cdots \quad \text{(i)}$$

$$Qi_1 - Si - Ni_2 = 0 \quad \cdots \quad \text{(ii)}$$

or
$$Ri + Mi_2 = Pi_1$$
$$Si + Ni_2 = Qi_1$$

or
$$\frac{Ri + Mi_2}{Si + Ni_2} = \frac{P}{Q}$$

or
$$i(QR - PS) = i_2(MQ - NP) \quad \cdots \quad \text{(iii)}$$

If $\frac{M}{N} = \frac{P}{Q}$, $\quad MQ - NP = 0$

So from equation (iii), $QR - PS = 0$

or $\frac{P}{Q} = \frac{R}{S}$

The condition for balance becomes $\frac{M}{N} = \frac{P}{Q} = \frac{R}{S}$,

So $S = \frac{QR}{P}$.

In practice, for construction of standard low resistances and determination of specific resistance P-Q and M-N are obtained from two pieces of resistance wires (AFB, CLD) whose electric mid-points are joined with the galvanometer terminals. As such P/Q becomes equal to M/N. S may be a wire whose length between D-B is adjusted by trial for balance. For balanced condition the resistance S of the length of the wire between the two connecting points at D and B is equal to the resistance of R, a standard resistance. In comparison experiments the final adjustment is made by shunting S (having a fixed resistance) by a resistance box. If the exact balance is obtained with a resistance S_1 in parallel with S, then

$$\frac{S.S_1}{S + S_1} = \frac{Q}{P}.R, \quad \text{thus } S \text{ is calculated.}$$

P, *Q*, *M*, *N* should be of the same material and be placed near to each other to ensure similar temperature variation, if there be any, in both of them.

In this arrangement the current leads with R and S are in series with the connecting rod (which has no effect on the balance condition) and the potential leads are in series with M, N ·· etc., which are wires of larger resistance. Hence the effect of the resistances of the junctions are of no consequence on the null condition.

Illustrative Examples

1. *If in a Wheatstone bridge circuit* $P=Q=25$ *ohms,* $R=100$ *ohms and* $S=105$ *ohms, calculate the current through the galvanometer (resistance* 50 *ohms) if the emf of the battery (of negligible resistance) is* 2 *volts.*

Solution : Applying Kirchhoff's law :

In EABC, $25p+25(p+g)=2$ ·· (i)

In EADC, $100r+105(r-g)=2$ ·· (ii)

In ABD, $25p-50g-100r=0$ ··(iii)

From (i) $p=\dfrac{2-25g}{50}$ ··· (iv)

From (ii) $r=\dfrac{2+105g}{205}$ ··· (v)

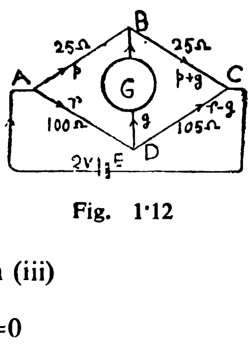


Fig. 1·12

Substituting these values of p and r in (iii)

$$\frac{2-25g}{2}-50g-\frac{200-10500g}{205}=0$$

$$\text{Hence } g=\frac{2}{9375} \text{ amp.}$$

2. *Find the current through the unbalanced Wheatstone bridge having* $P=1$ *ohm,* $Q=2$ *ohms,* $S=3$ *ohms and* $G=4$ *ohms, the battery being of emf* 2 *volts and internal resistance* 2 *ohms.*

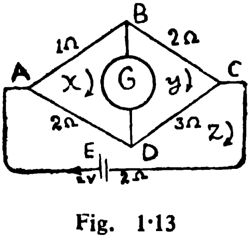


Fig. 1·13

Solution : Considering the Maxwell-current in the respective circuits as x, y, z and applying Kirchhoff's law in the modified form,

In ABD, $x+4(x-y)+2(x-z)=0$

or $7x-4y-2z=0$ ··· (i)

In BCD, $2y+3(y-z)+4(y-x)=0$

or $-4x+9y-3z=0$ ... (ii)

In ADCE, $2z+2(z-x)+3(z-y)= 2$

or $2x+3y-7z=-2$ ... (iii)

Solving (i) and (ii),

$$\frac{x}{12+18}=\frac{y}{8+21}=\frac{z}{63-16}$$

$$\text{Hence} \quad x=\frac{30}{47}z, \quad y=\frac{29}{47}z$$

Substituting these values in (iii),

$$\left(-\frac{60}{47}-\frac{87}{47}+7\right)z=2, \quad \text{or} \quad z=\frac{47}{91}$$

Current through the galvanometer is

$$x-y=\left(\frac{30}{47}-\frac{29}{47}\right)\frac{47}{91}=\frac{1}{91} \text{ amp.}$$

3. *In a balanced Wheatstone bridge the resistances are* $P=100$ *ohms,* $Q=10$ *ohms and* $R=20$ *ohms. Find S and the current drawn from the battery of emf 4 volts and internal resistance 2 ohms.*

Solution : $S=\frac{Q}{P}.R=\frac{10}{100}\times 20=2$ ohms.

In a balanced condition $(P+Q)$ and $(R+S)$ may be considered to be in parallel circuit with the battery. Hence the equivalent resistance of P, Q, R, S is obtained as

$$\frac{1}{r}=\frac{1}{P+Q}+\frac{1}{R+S}=\frac{1}{110}+\frac{1}{22}=\frac{6}{110}$$

$$\text{or} \quad r=\frac{110}{6} = 18\frac{1}{3} \text{ ohms.}$$

The required current is $i=\frac{4}{\frac{55}{3}+2}=\frac{12}{61}$ ampere.

1-6. APPLIANCES AND APPLICATIONS

Shunt : The principle of diverting a part of a current flowing through a resistor by joining another resistor in parallel

with it is utilised in an appliance known as shunt. Two such resistances in parallel act as shunt to each other. If the shunt has relatively lower resistance, greater part of the main current flows through it. Let R_1 and R_2, two resistances in parallel, be joined to an *emf* E. The equivalent resistance of the circuit (fig. 1·14a) is given by

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

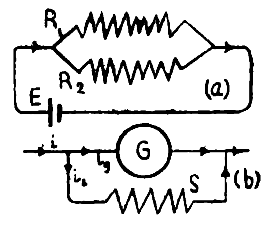


Fig. 1·14

The current supplied by the cell is

$$i = E.\frac{R_1 + R_2}{R_1 R_2}$$

The currents flowing in the two branches, denoted by i_1 and i_2 respectively, are obtained by considering the potential difference at the terminals,

$$E = iR = i.\frac{R_1 R_2}{R_1 + R_2}$$

Again $E = i_1 R_1$, hence $i_1 = \frac{E}{R_1} = i.\frac{R_2}{R_1 + R_2}$

and $E = i_2 r_2$, hence $i_2 = \frac{E}{R_2} = i\frac{R_1}{R_1 + R_2}$

So $\frac{i_1}{i_2} = \frac{R_2}{R_1}$; the currents in the two branches are inversely proportional to the respective resistances—the greater the resistance, the smaller is the current flowing through it.

GALVANOMETER SHUNT: In order to control the current flowing through a galvanometer, a resistance is joined in parallel with the galvanometer coil as its shunt. If G and S are respectively the galvanometer and the shunt resistances then the current through the galvanometer (fig. 1·14b) expressed as a fraction of the main current (i) is obtained as

$$i_g = i.\frac{S}{S+G}$$

If $\frac{1}{n}$th of the total current is intended for the galvanometer, the necessary value of the shunt is obtained from the equation written below :

$$\frac{S}{S+G}=\frac{1}{n} \quad \text{or} \quad S=\frac{1}{n-1}.G$$

The factor $\frac{S+G}{S}$ by which the galvanometer current i_g is to be multiplied to obtain the total current is called the *multiplying power* of the shunt.

Galvanometers are often shunted to save the apparatus from damage due to heavy currents. But it would be realised from the foregoing expression that each galvanometer according to its coil resistance (G) must have its own set of shunts if the desired fraction of a current is to be passed through the galvanometer. This difficulty has been overcome in a specially designed type of shunt-box suitable for use with any galvanometer.

Ayrton Mather Universal Shunt : It consists of a high resistance R having several tapping points at some fractional values of R. Usually the total value of R is 10000 ohms and tapping points are for 1000, 100 and 10 ohms. The whole of R or $\frac{1}{n}$th fraction ($\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$) of it can be put in parallel with any galvanometer coil. But whatever resistance be put as shunt, the total resistance in the shunt and the galvanometer circuits taken together remains constant. As shown in the sketch (fig. 1·15) the galvanometer is connected at T_1 and T_2. The current entering at one end of the resistance coils may leave at the other end or at some tapping point through T_3. If the full value of R is in parallel with the galvanometer coil of resistance G, the current flowing through the galvanometer is given by

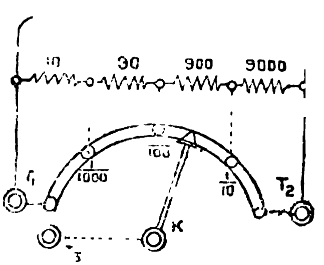


Fig. 1·15

$$i_g = i.\frac{R}{G+R}$$

Where i is the total current from source of supply. If R is high in comparison with G, $i_g \rightarrow i$. Let the sliding contact be shifted to any other point so that $\frac{1}{n}$th of R is in parallel with the galvanometer and the rest of R *i.e.* $(R-R/n)$ is in series with the galvanometer. Under such an arrangement the fraction of the total current flowing through the galvanomoter is given by

$$i'_g = i.\frac{R/n}{\frac{R}{n}+\left(G+R-\frac{R}{n}\right)} = \frac{1}{n}\cdot\frac{iR}{G+R} = \frac{1}{n}\cdot i_g$$

This value i'_g is $\frac{1}{n}$th of the current flowing through the galvanometer when full value of R is in parallel with the galvanometer. Thus by connecting one of the terminals of the source to one end of R and the other to a tapping point on R, the current, may be reduced to $\frac{1}{10}$, $\frac{1}{100}$ or $\frac{1}{1000}$ fraction of the maximum current that is practically of the main current. This effect is obtained with any galvanometer irrespective of its resistance.

Post Office Box : A practical and ready method of utilising the principle of Wheatstone bridge for comparison of ordinary resistances is provided for in a special type of resistance box, known as Post Office Box. In this box there is an assembly of different sets of plug type resistors arranged in three branches to form the three arms of a Wheatstone bridge.

As shown in the diagram, the resistances in the branches AB, BC are meant for ratio-arms resistances. Each of these sets contains coils of resistances 10, 100 and 1000 ohms. The third arm resistance is obtained from AD branch which contains a number of coils so that by suitable plucking of plugs any resistance from 1 to 11110 ohms may be put in this branch. The unknown resis-

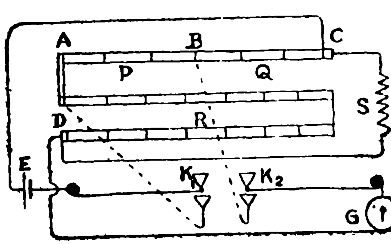


Fig. 1·16

tance meant for determination is connected between C and D. The battery (placed between A-C) and the galvanometer (placed between B-D) connections are closed through tapping keys K_1 and K_2 respectively. This mode of closing the circuit is to avoid unnecessary heating of the coils, the current being passed momentarily when necessary by pressing the keys. The battery circuit is to be closed always before the galvanometer circuit to avoid a throw in the galvanometer coil which may be produced by electromagnetic induction even when the bridge is balanced for steady current. The third arm resistance (R) which gives the balance is to be multiplied by the ratio of the resistances (Q/P) put in the ratio-arms for obtaining the value of the unknown resistance.

Metre Bridge : For measurement of resistance by applying the principle of Wheatstone bridge with better precision than that may be obtained with a P. O. Box, the Metre bridge is used.

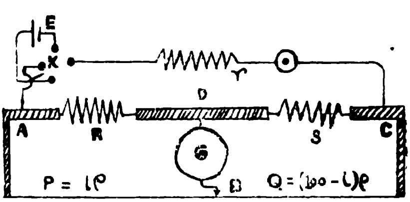


Fig. 1·17

Three copper strips A, D, C are so mounted on a wooden base as to leave three gaps, one of them being one metre long. In this gap a straight uniform resistance wire of one metre length is stretched over a scale lying below the wire. The other two small gaps between A-D and D-C are meant for insertion of two resistors that are to be compared. One of these is of known value and the other is the unknown resistor whose value is to be determined. By a sliding contact arrangement any point on the resistance wire may be put in connection with the galvanometer having one of its terminals connected between the two resistors mentioned. A battery is connected at the ends of the wire.

The connections, as shown in the diagram (Fig. 1·17), form a Wheatstone net circuit. If through trials a balance is obtained by making a contact of the slider at a distance l from

one end of the wire and if ρ be the resistance per unit length of the wire, then

$$\frac{R}{S}=\frac{P}{Q}=\frac{l\rho}{(100-l)\rho}$$

$$or\ \ S=R.\frac{100-l}{l}$$

If R is known, S may be calculated.

The procedure is subject to error due to (*i*) non-uniformity of the wire, (*ii*) themo-electric effect, (*iii*) end effects and (*iv*) zero-error of the scale. These are to be eliminated. Zero-error is corrected by interchanging the position of the two resistors and taking the mean value of the calculated value of S in two positions. The thermoelectric effect is eliminated by obtaining the null points by reversal of current and taking the mean length as the corrected balancing length. The other two errors can be corrected for by separate experiments, *viz.* end corrections and calibration of the bridge wire.

GREATEST ACCURACY : Any discrepancy in the determination of the length of the wire giving the balance brings some error in the calculated value for the unknown resistance. The relative error varies for null-points obtained at different lengths. Consider dS to be a small increase in the calculated value of S for a change dl in the length of the wire corresponding to the balance point. The value of S is related to l, the length of the wire, by the following equation

$$S=R.\frac{100-l}{l}=\frac{100R}{l}-R$$

By differentiation, $dS = -R.\frac{100}{l^2}\,dl$

$$\text{So }\frac{dS}{S}=-R\frac{dl}{l\,(100\text{-}l)}$$

$\frac{dS}{S}$ is the relative error in S and for greatest accuracy this should be minimum. This is obtained when $(100-l)l$ is maximum. So for greatest accuracy

$$\frac{d}{dl}[\,(100-l)\,l\,]=0$$

$$\text{or} \quad \frac{a}{dl}(100l - l^2) = 0$$

$$\text{or} \quad 100 - 2l = 0$$

$$\text{or} \quad l = 50$$

This indicates that for greatest accuracy the null point should be near the midpoint of the wire. This is obtained when R and S are nearly equal.

END CORRECTIONS : There may be error due to small resistances of contact at the ends of the wire and out of the fact that the wire may not be exactly one metre long. Such errors are corrected by a method due to Ferguson.

Let α_1 and α_2 represent the errors at the two ends respectively expressed as centimeters of length to be added to the bridge wire length on the two sides of the null point. If R_1 and R_2 are two known resistances placed in the gaps and l_1, l_2 the lengths into which the bridge wire is divided for a balance, then

$$\frac{l_1+\alpha_1}{l_2+\alpha_2} = \frac{R_1}{R_2} = x \text{ (say)}$$

$$\text{So} \quad xl_2 - l_1 = \alpha_1 - \alpha_2 x$$

Writing $xl_2 - l_1 = y$, the above relation may be obtained in the form $y = \alpha_1 - \alpha_2 x$. Taking different values of R_1 and R_2, the corresponding values of x and y are obtained. x-y are plotted to obtain the graph of the equation $y = \alpha_1 - \alpha_2 x$. α_1 is obtained from the intercept on the y-axis of the straight line representing the equation and α_2 is measured from the slope of the straight line.

Carey-Foster's bridge : The difference in magnitudes of resistance of two resistors of nearly equal values may be accurately obtained by means of a modified form of metre bridge due to Carey-Foster.

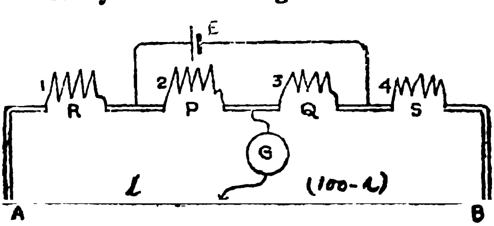


Fig. 1·18

The bridge contains four gaps and in the two extra gaps at

the two ends, two standard resistances are inserted so as to be in series respectively with the two portions of the bridge wire on either side of the null point. The arrangement is as shown in fig. 1·18. R and S, the resistances to be compared, are in the two outer gaps. P and Q are two nearly equal resistances placed in the inner gaps and these form the ratio-arms of Wheatstone bridge. R and S are in series respectively with the portions of the bridge wire adjacent to each of them and separated by the null point. These combinations form the other two arms. For a balanced bridge. we may write,

$$\frac{P}{Q}=\frac{R+(l_1+\lambda_1)\rho}{S+(100-l_1+\lambda_2)\rho} \quad \cdots \quad \cdots \quad \text{(i)}$$

l_1 is the length of the wire from the end to which R is joined, λ_1 and λ_2 are the end corrections at the respective ends and ρ is the resistance per unit length of the bridge wire.

Let the positions of R and S be interchanged and a new balance point be obtained at a length l_2, then

$$\frac{P}{Q}=\frac{S+(l_2+\lambda_1)\rho}{R+(100-l_2+\lambda_2)\rho} \quad \cdots \quad \cdots \quad \text{(ii)}$$

Adding the numerator and denominator of the right hand side in each of the equations (*i.e.* in i and ii) and rewriting the equations with the new numerators obtained by summation, the denominators remaining as before, we get

$$\frac{R+S+(\lambda_1+\lambda_2+100)\rho}{R+(100-l_2+\lambda_2)\rho}=\frac{R+S+(\lambda_1+\lambda_2+100)\rho}{S+(100-l_1+\lambda_2)\rho}$$

Hence $R+(100-l_2+\lambda_2)\rho=S+(100-l_1+\lambda_2')\rho$

or $R-S=(l_2-l_1)\rho$

This expression may be utilised in obtaining the value of ρ and in calibrating the bridge wire.

TO DETERMINE ρ : A standard 1-ohm coil is taken to represent R, S is obtained by a similar 1-ohm coil shunted by a standard 10-ohm coil, whence S becomes 0·909 ohm. If l and l' are the readings for the two null points when the bridge is used as described, then

$$\rho\,(l-l')=1-0{\cdot}909=0{\cdot}091 \text{ ohm}$$

ρ is calculated from this equation.

TO CALIBRATE THE BRIDGE WIRE : The bridge wire may not have uniform resistance per centimetre throughout its entire length. For this it is often necessary to divide the bridge wire into a number of small lengths (which may not be equal) of equal resistance. This may be done by the foregoing method and obtained from the equation $(R-S)=(l-l')\rho$. If the wire be not uniform, $(l-l')$ will be different for null points at different positions along the length of the wire. This is obtained by suitable adjustments of P and Q, keeping R and S constant. Let these be l_1, l_2, l_3 etc. from one end to the other. The mean of these lengths is calculated and the difference of each length from the mean is obtained. These give the corrections at different positions of the wire. To apply the correction necessary for a length obtained for a balance, these errors are to be algebraically added from one end up to the relevant point of balance.

Callendar and Griffith's bridge : This is another modification of Wheatstone bridge used to measure the change in resistance of the coil in a platinum thermometer.

In this bridge a P. O. box is used in combination with a metre bridge. The circuit is shown in fig. 1·19. The ratio-arms P-Q of the P. O. box are set to equality and the platinum coil (resistance S) with its leads (resistance T) is connected with the box terminals so as to form the fourth arm. The third arm resistance obtained from the box is in series with the compensating leads (resistance C) of the thermometer. The third and the fourth arms are connected through the bridge wire (AB). The sliding key of the bridge is connected with the junction of the ratio-arms through the galvanometer. The battery is connected as usual. A suitable resistance (R) is to be inserted in the third arm and a null point is to be obtained by sliding the jockey on the bridge wire.

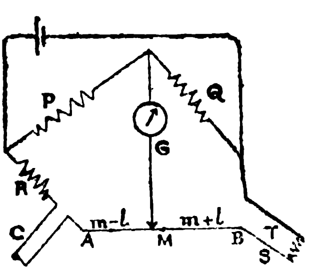


Fig. 1·19

Let the balance be obtained on placing the slider at a distance x (towards R) from the electric mid-point (M) of the bridge wire. The mid-point is at a distance, say, m from the R-end of the wire. If R is the resistance in the third arm of the box and ρ is the resistance per unit length of the bridge wire, then for balance

$$R+C+(m-x)\rho=S+T+(m+x)\rho$$

$$\text{Since} \quad C=T, \text{ hence } S=R-2\rho x$$

If l is the distance of the null point from R-end, then $x=(m-l)$; hence $S=R-2(m-l)\rho$.

To determine ρ, R is increased by one ohm. If this shifts the null point through a distance l_0, then

$$2l_0\rho=1, \quad \text{hence} \quad \rho=\frac{1}{2l_0} \text{ ohm.}$$

Determination of Galvanometer Resistance : In the arrangement known as *Kelvin's method*, a P. O. box is used and the galvanometer, whose resistance is to be measured, is placed in the fourth arm. In the normal position of the galvanometer in the Wheatstone bridge there is a short circuiting key ; this is obtained by joining the P-Q and R-S junctions of the box by a conducting wire through the relevant box key. With such an arrangement if the battery circuit is closed there will be a deflection in the galvanometer. This deflection is brought within the scale either by adjusting the current or brought to zeroreading by twisting the suspension in the case of a suspended coil type galvanometer.

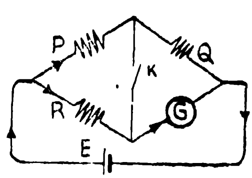


Fig. 1·20

Now R is so adjusted that neither pressing nor releasing of the short circuiting key causes any change in deflection. For a balanced bridge the current in any of the four arms is independent of the galvanometer circuit resistance. So under the condition stated, the, galvanometer resistance is obtained from the equation $G=QR/P$.

Determination of Battery resistance : In a modified arrangement of Wheatstone bridge (known as *Mance's method*) the battery is placed in the fourth arm and the normal connecting points of the battery are short-circuited through the relevant tap-key (K). Under a balanced condition change of the *emf* of the battery does not affect the current condition of the galvanometer circuit. The resistances P, Q, R are so adjusted that the galvanometer deflection remains unaltered by pressing or releasing of the key K. This means that if a current passes through K, it does not affect the galvanometer current. This is possible only when the null condition $(P/Q=R/S)$ is satisfied. Hence we know under the conditon stated $P/Q=R/B$ and B is calculated.

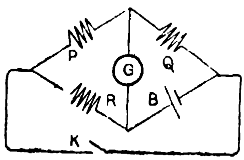


Fig. 1·21

This method has some draw-backs. A current flows through the galvanometer which may be too large during testing. Further, current flowing through the battery is likely to alter its resistance.

Principle of Potentiometer : If a current is passed through a uniform resistance wire, there is a uniform drop of potential through it along its length and the potential difference between any two points on it is proportional to the distance separating them. Suppose PQ is such a wire having a supply of current from a source of *emf* B. If i be the current through the wire, the potential drop over a length l of the wire is $il\rho$, ρ being the resistance per unit length. Consider that one terminal of another cell of *emf* E is so connected to P that it tends to send a current through PQ in the same direction as B. The other terminal of E is connected to a contact key sliding on PQ, through a galvanometer G. If $B>E$, the potential difference between P-Q will be normally greater than E. The potential difference between P-Q may be diminished by intro-

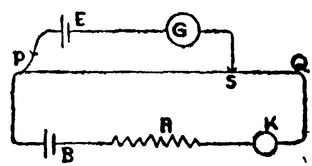


Fig. 1·22

ducing some external resistance in the circuit of the cell B (usually called the ***driver cell***). If this resistance is adjustable, the potential drop between P-Q may be arranged to be of any value greater or less than E. If the potential difference between P-Q is not less than E, a point S on PQ may be so found that the potential difference between P-S is equal to E. In such a case no current will flow in the galvanometer if the sliding key makes a contact at S. If $V_A - V_B$ be greater or less than E, a current flows through the galvanometer in either case, but in mutually opposite directions. Such an arrangement forms a ***potentiometer***, which may be used for comparison and determination of *emf*s, comparison of low resistances and for measurement of current.

COMPARISON OF EMF's : As stated in the foregoing paragraph, let a cell of *emf* E_1 be balanced at a length l_1 of the potentiometer wire and a cell of *emf* E_2 at a length l_2 under the same condition existing in the driver cell circuit. Since the *emf* of a cell when balanced over a potentiometer wire is proportional to the balancing length, so $E_1/E_2 = l_1/l_2$.

DETERMINATION OF E.M.F. : Let i be the current through a potentiometer wire and ρ be its resistance per unit length. If a cell of *emf* E is balanced at a length l of the wire then $E = il\rho$.

In order to make the arrangement more sensitive the current through the wire is kept as low as possible, so that the balancing length is increased. This may be done by an adjustable resistance joined in series with the potentiometer wire. But if by such an adjustment i becomes too low, the total drop of potential $iL\rho$, over the entire length (L) may be less than the *emf* of the cell under measurement and a balance point will not be obtained in such a case.

MEASUREMENT OF CURRENT : Let a low resistance of known value r be included in the circuit, the current in which is to be measured. The positive terminal of the resistor is joined to the positive end of the potentiometer wire (both of these may be negative as well) and the other terminal of the

resistor is connected to the slider through a voltage sensitive galvanometer. If i be the current through the resistance r, the potential drop between its ends is ir. Let this be balanced at a length l of the of the potentiometer wire carrying a current I. If the resistance per unit length of the potentiometer

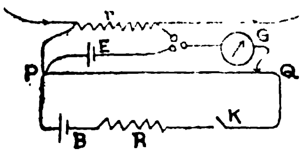


Fig. 1·23

wire be ρ, then

$$ir = Il\rho = Il.\frac{R}{L}$$

R is the resistance of the potentiometer wire of length L.

$$\text{So } r = \frac{I}{i} \cdot \frac{l}{L} . R$$

I may be measured with an ammeter inserted in the driver cell circuit or calculated from a knowledge of the *emf* of the driver cell and the total resistance in the circuit.

Alternatively, a standard cell of emf E (as shown in fig. 1·23) may be balanced under similar conditions and if the balancing length in this case is l_0, then

$$\frac{ir}{E} = \frac{l}{l_0} \quad \text{or} \quad i = \frac{E}{r} \cdot \frac{l}{l_0}$$

This method gives an accurate value of i.

COMPARISON OF LOW RESISTANCES : If the same current i flows through two resistances r_1 and r_2, the potential drops across each of them (ir_1 and ir_2) are proportional to their resistances. Hence a comparison of potential drops provides a method for comparison of two low resistances. This may

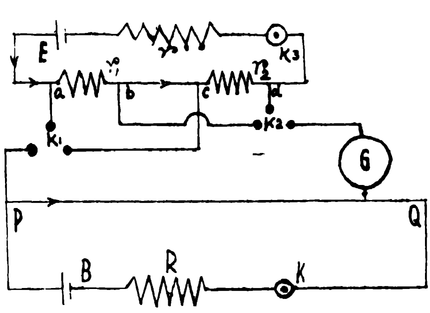


Fig. 1·24

be done with a potentiometer. A circuit is made with the two resistances under investigation in series with a cell. A poten-

tiometer is set up with proper arrangement to connect the two terminals of the two resistances in turn with the potentiometer circuit, (fig 1·24). The lengths of the potentiometer wire balancing the potential drops at the ends of the two resistances are obtained in separate observations but under the same condi tions existing in the driver circuit. If l_1 and l_2 are the lengths balancing the potential drops ir_1 and ir_2 respectively, then

$$\frac{ir_1}{ir_2}=\frac{l_1}{l_2}, \text{ or } \frac{r_1}{r_2}=\frac{l_1}{l_2}$$

Note : When a potential drop across a resistor due to a current flowing through it is to be measured, it is to be provided with two pairs of terminals *current and potential leads.* Through one pair (C_1, C_2 in fig. 1·25) the current enters and leaves and by connection with the other pair (P_1, P_2) the potential difference at the ends of the resistance of particular value is measured. The resistance considered in potential drop measurements is actually the resistance encountered by the current between the potential leads. Hence in standard resistances the potential leads are to be accurately joined between the points giving the exact resistance specified.

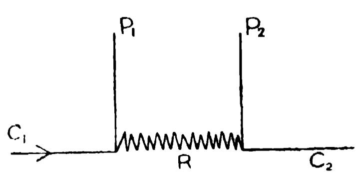


Fig. 1·25

MEASUREMENT OF BATTERY RESISTANCE : The *emf* of the battery is balanced against a length l_1 of the potentiometer wire when there is no resistance across the battery, in other words, when the battery is off load. A known resistance R is then placed across the battery terminals and let a balance be obtained under this arrangement at a length l_2.

Where the battery is loaded, the potential drop (V) across the load is given by $V=E-iR_B$ where i is the current through the load and R_B is the battery resistance.

Again $i = V/R$, so

$$V = E - \frac{V}{R} . R_B$$

$$\text{or} \quad R_B = \frac{E - V}{V} . R$$

Since E and V are proportional to l_1 and l_2, so

$$R_B = \frac{l_1 - l_2}{l_2} . R$$

EXERCISES ON CHAPTER I

I-1. Discuss the process of flow of electric current through a solid conductor. Obtain an expression for the resistance of the conductor from the idea of drift of charge through it due to an electric field.

If I be the current density in a conductor, show that div $\mathbf{I}=0$, if there be no source or sink in the circuit.

I-2. State and explain Kirchhoff's law for the distribution of current in a network of conductors.

A 4-volt battery with 4 ohms resistance in series is joined in parallel with a 8-volt battery having 12 ohms resistance in series. The combination is applied to send a current through a load of resistance 8 ohms. Calculate the currents in all the three branches.

[*Ans* : $\frac{9}{11}$, $\frac{8}{11}$, $-\frac{1}{11}$ amp]

I-3. State the laws governing the distribution of current in a network of conductors.

Four wires each having 1 ohm resistance are joined one after another in a quadrilateral symmetry. The diagonal points are joined by two cells respectively, each of *emf* 1 volt and 2 ohms internal resistance. Show that no current flows through one pair of conductors forming the opposite sides of the quadrilateral.

I-4. Twelve similar wires each of resistance 1 ohm are so joined as to form a cube. If the terminals of a source of *emf*

are joined to the two adjacent corner points of the cube, calculate the equivalent resistance of all the conductors in the cube.

Hints : The distribution of current is shown in fig. 1·26. It is obtained by considering the pairs of paths (*AEFB*, *ADCB*). (*AE*, *FB*) and (*AD*, *CB*) as similar. Solve for i_1, i_2 and i_3 by applying Kirchhoff's law in meshes *AEFB*, *CDHG* and Ohm's law in the resistor *AB*. Obtain the main current. The equivalent resistance is equal to applied *emf*/main current.

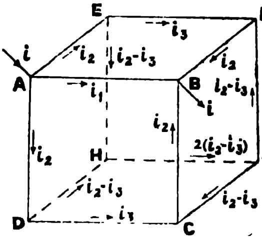


Fig. 1·26

[*Ans* : $\frac{7}{12}$ ohm]

I-5. Obtain the best arrangement of a number of cells grouped to form a battery. What is meant by 'lost volt' in a battery ?

A battery of *emf* 6 volts and 0·5 ohm internal resistance is joined in parallel with another battery of *emf* 10 volts and internal resistance 1 ohm. Calculate the current through an external load having a resistance 12 ohms and obtain also the currents in the respective battery circuits.

[*Ans* : 0·595, 2·865, − 2·27 amp]

I-6. An accumulator is to be charged from a 200-volt mains. The charging current necessary is 4 amps. Show how would you use several lamp resistances (200 volt-100 watt) to obtain the requisite current from the supply mains.

[*Ans* : 8 lamps in parallel joined in series with the mains]

I-7. Three cells having the following specifications are joined in parallel. The middle one is in opposition to the other two. Calculate the current through each cell and the terminal voltage.

Cell *A* : 2 volt—4 ohm, Cell *B* : 1 volt—3 ohm, Cell *C* : 4 volt—2 ohm. [*Ans* : $\frac{10}{19}$, $\frac{9}{19}$, $\frac{19}{19}$ amps ; $\frac{14}{19}$ volt]

I-8. Find the condition for balance in a Wheatstone's net and obtain an expression for the current through the galvanometer when the bridge is out of balance.

I-9. Give the theory of the Wheatstone bridge and deduce the condition for having the most sensitive arrangement of placing the battery and the galvanometer. Discuss the difficulty with such an arrangement meant for best sensitvity.

I-10. In a Wheatstone bridge arrangement the four resistances in the four arms AB, BC, CD, DA are respectively 1000, 10, 4, 50 ohms. The resistance of the galvanometer connected across BD is 20 ohms. Calculate the current through the galvanometer when a potential difference of 10 volts is maintained across AC. [*Ans* : $\frac{5}{874}$ amp.]

I-11. Discuss the best value of resistors in the arms of a Wheatstone bridge giving the maximum sensitivity. Is the value of the galvanometer resistance in any way important in this respect ?

I-12. Give the theory underlying the working of a Kelvin double bridge. What are its advantages ?

I-13. What is a shunt ? Describe the principle underlying the working of Ayrton-Mather universal shunt. What are its advantages ?

I-14. Why is a Post Office box unsuitable for measuring low resistance ? Give the theory of the potentiometer and show how it may be used for the comparison of two low resistances.

I-15. What are the sources of error in measurement with a metre bridge ? How are they eliminated ? What is the advantage of having a null point in the mid-portion of the wire ?

I-16. Discuss the theory of a Potentiometer. State how you can measure current with a potentiometer.

I-17. Describe Callendar and Griffth's bridge and its application in obtaining the resistance of a platinum thermometer.

I-19. Explain how a potentiometer may be used to obtain the resistance of a battery.

I-20. Discuss the effect of (a) temperarure, (b) incident light, (c) magnetic field on the resistance of a conductor.

CHAPTER II

ELECTRO-MAGNETISM

II-1. MAGNETIC EFFECT OF CURRENT

Laplace's Law : This law was formulated about 1820 as a working hypothesis and may be stated as follows :

The intensity of magnetic field (δH) *produced by a short segment* (δs) *of a conductor carrying current* i *at a point at a distance* r *in a direction* θ *is found to be related to the different factors as shown below,*

$$\delta H \propto \frac{i.\delta s. \sin\theta}{r^2}$$

$$\text{or} \quad \delta H = k.\frac{i.\ \delta s.\ \sin\theta}{r^2}$$

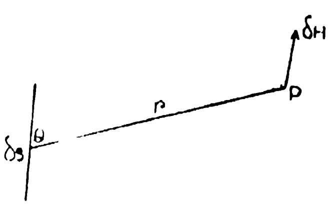


Fig. 2 1

k being considered as a constant.

The direction of the field is at right angles to the plane containing the element and the point considered. Its sense being determined by the cork-screw or thumb rule.

By suitably fixing a definition of unit current measured on the basis of its magnetic effect the constant k in the above equation may be made equal to 1, and if the strength of the current is measured by this unit (electro-magnetic unit defined hereafter), the magnetic field produced by a current element may be calculated as

$$\delta H = \frac{i.\ \delta s.\ \sin\theta}{r^2}$$

$$\text{or} \quad \delta H = \frac{i}{r^3}[\ \delta s \times r\ ]$$

This relation showing the magnetic effect of current is also known as *Bio-Savart's law.*

Electromagnetic units : Based on the principle of measurement of current by its magnetic effect, there is a system of

electrical measurement called e. m. units. The fundamental units are defined below.

E. M. UNIT OF CURRENT is defined as that current which when flowing through a conductor one centimetre in length bent into an arc of a circle of one centimetre radius produces a magnetic field of intensity one oersted at the centre of the circle.

AMPERE, the practical unit of current, is one-tenth of e. m. unit current. It is called *true ampere*.

E. M. UNIT OF CHARGE is measured as the charge which passes in one second through any cross-section of the conductor carrying one e. m. unit current.

COULOMB is one-tenth of e. m. unit charge and is actually the charge carried by one ampere in one second.

E. M. UNIT OF POTENTIAL : Two points in an electric circuit are said to have unit (e. m. u.) potential difference if in the transference of one e. m. unit of charge between then the energy involved is one erg.

If in the transference of one coulomb of charge between two points in an electric circuit the energy involved is one joule, the potential difference between the points is said to be one *volt* (true volt). This is the practical unit of potential.

1 joule$=10^7$ ergs and 1 coulomb$=10^{-1}$ e. m. u. of charge, hence 1 volt$=10^8$ e. m. units of potential.

Ampere's theorem : Ampere in 1823 formulated a law concerning the fact that a steady current flowing in a closed circuit produces the same magnetic field as a magnetic shell of certain specifications. *Ampere's theorem of equivalent magnetic shell* is stated as follows :

Every conductor carrying a current (so far as its magnetic effect is concerned) is equivalent to a magnetic shell whose bounding edge coincides with that of the conductor, the strength of the shell (its magnetic moment per unit area) being proportional to the strength of the current.

If σ is the strength of the equivalent shell and i is the stren-

gth of the current, then according to Ampere's theorem $\sigma = ki$. *By defining the e. m. unit current as the current which produces at any point the same magnetic field as is produced by a magnetic shell of unit strength and of same boundary as the conductor,* we may write $\sigma = 1$, k being 1.

It should be noted that the two definitions of e. m. unit current, one from Laplace's law, the other as realised by Ampere's theorem, actually refer to the current of same strength and as such the two units are identical.

Equivalence of a magnetic shell and an electric circuit ls restricted by two provisios : (i) the point at which the field is to be obtained must not be inside the material of the shell and (ii) in a magnetic medium of permeability μ, the equivalence is to be obtained as $\sigma = \mu i$.

Equivalence is further bounded by the rule that if to an observer the current in the conductor appears to flow *anti-clockwise*, he will consider himself facing the north pole of the shell.

Circuital form of Ampere's theorem : Since a conductor carrying current is associated with a magnetic field around it, work is necessary to carry a magnetic pole round the conductor. Ampere's circuital theorem expresses the magnetic effect of current in terms of this work. This is also called as the circuital form of Laplace's law.

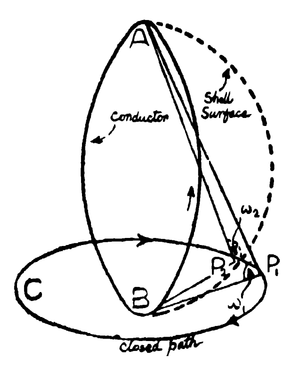


Fig. 2·2

Let AB be a closed circuit in which a steady current i flows. The surface of an equivalent shell is shown in broken line (fig. 2·2). Let C be a closed path linked with the circuit and P_1, P_2 are two points very close to one another respectively on the positive and negative sides of the shell. Let ω_1 and ω_2 be the solid angles subtended by the boundary of the shell at these points respectively. Then the magnetic potentials at P_1 and P_2 are $\Omega P_1 = i\omega_1$ and $\Omega P_2 = i\omega_2$.

Work done in going from P_1 to P_2 with unit magnetic n-pole measured as the increase in magnetic potential is

$$i\omega_1-(-i\omega_2)=i(\omega_1+\omega_2)$$

Let the points P_1 and P_2 approach each other. In such a case the sum of the solid angles AP_1B and AP_2B becomes more and more nearly equal to the solid angle subtended at a point by the whole of a sphere surrounding the point, that is to 4π. Hence when P_1 approaches P_2 to coincide, $(\omega_1+\omega_2)$ becomes equal to 4π. Therefore *the work done in carrying unit pole round a closed path linked once with the circuit carrying a current is* $4\pi i$. If the path is traversed n-times, the work amounts to $4\pi ni$. As such the magnetic potential due to a current becomes a multi-valued function. This is quite logical since threading by a magnetic pole changes the condition of the circuit and a new state of affairs appears every time the pole circulates the circuit.

Line integral of magnetic field : The work done in carrying unit pole along any path in a magnetic field from one point to another is called the line integral of the field between the points. If the strength of the field at any point of the path be H and θ its inclination to the path, the component of the force on a pole acting along the path is $H\cos\theta$ and the work done in carrying it along an infinitesimal distance ds is $H\cos\theta.ds$. So the work done in carrying unit pole between two points A-B is $\int_A^B H\cos\theta.ds$. If the path is a closed one linking once with the current i, this work as shown before, is $4\pi i$, hence

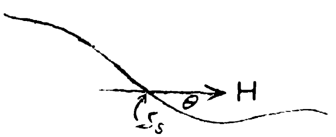


Fig. 2 3

$$\oint H\cos\theta.ds = 4\pi i$$

the symbol $\oint$. means line integral round a closed path. If the path is not linked with any current $\oint H\cos\theta.ds = 0$.

Expressed in vector notation, the line integral of a magnetic field round a closed path linked once with a current i is

$$\int_0 \mathbf{H}.\, d\mathbf{s} = 4\pi i$$

The line integral of a field round a closed path enclosing unit area inside the conductor is obtained as *curl of the field,* so if I be the current density

$$\text{Curl } \mathbf{H} = 4\pi I$$

II-2. MAGNETIC FIELD DUE TO A CONDUCTOR

Applications of Ampere's theorem and Laplace's law : The magnetic field due to conductors of different forms carrying current may be obtained by the application of either Ampere's theorem or Laplace's law. That these two have the same physical significance may be realised by solving a problem by applying the two methods separately and obtaining the same result.

Magnetic field due to a linear current : Consider a straight infinitely long conductor carrying i (*e. m. u.*). We shall obtain the field at any point P, at a distance r from the conductor.

BY AMPERE'S THEOREM : By symmetry the magnetic field has the same magnitude at all point on a circle of radius r passing through P (fig. 2·4) having its centre on the axis of the conductor. If H be the field at P, then bv circuital theorem, remembering that ds is everywhere at right angles to H, we have

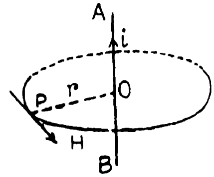


Fig. 2·4

$$4\pi i = \int_0 H.\, ds = H\int_0 ds = 2\pi r H$$

$$\text{Hence} \quad H = \frac{2i}{r}$$

Let the conductor be of radius a. If the point at a distance r from the axis of the wire is so that $0 < r > a$, the current linked with the circular path through a is ir^2/a^2. Hence by Ampere's circuital theorem,

$$\int_0 H.ds = 2\pi r H = \frac{4\pi i r^2}{a^2}$$

$$\text{or} \quad H = \frac{2i}{a} . r$$

BY LAPLACE'S LAW: Let AOB be a linear conductor carrying a current i and ab any portion of it of length δl. The field due to this element at a point P at a distance x from it where the direction of P makes an angle θ with ab, is given by

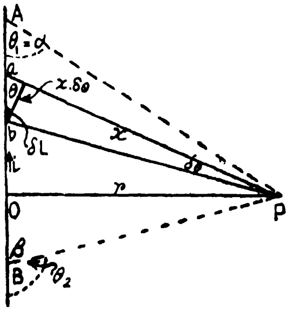


Fig. 2·5

$$\delta H = \frac{i.\ \delta l.\ \sin\theta}{x^2}$$

As shown in the diagram (fig. 2·5) if $Oa = l$, $ab = -\delta l$, $OP = r$, the perpendicular distance of AOB from P, $l = r\cot\theta$, $x = r$ cosec θ and $\delta l = -x.\ \delta\theta.$ cosecθ $= r\ \text{cosec}^2\theta.\delta\theta$

$$\text{Hence} \quad \delta H = -\frac{ir\ \text{cosec}^2\theta.\delta\theta.\ \sin\theta}{r^2\ \text{cosec}^2\ \theta} = -\frac{i}{r}\ \sin\theta.\delta\theta.$$

For a conductor of finite length if P makes an angle θ_1 and θ_2 with the ends of the conductor, by integration

$$H = -\frac{i}{r}\int_{\theta_2}^{\theta_1} \sin\theta.\ d\theta = \frac{i}{r}\Big[\cos\theta_1 - \cos\theta_2\Big]$$

Writing $\alpha = \theta_1$ and $\beta = \pi - \theta_2$, as shown in the sketch,

$$H = \frac{i}{r}\Big[\cos\alpha + \cos\beta\Big]$$

For an infinitely long conductor $\alpha = \beta = 0$, hence $H = \frac{2i}{r}$.

Magnetic field due to a circular current: Field at any point on the axis of a circular coil of radius a and carrying a current i (*e.m.u*) is obtained as follows:

EQUIVALENT SHELL METHOD: Since the magnetic potential at any point due to a shell of strength ϕ, subtending

a solid angle ω there, is given by $V=\phi\omega$, so the magnetic potential due to a closed circuit through which a current i is flowing will be $V=i\omega$. In particular position with respect to the circuit, ω can be calculated geometrically.

The solid angle subtended by a circle at a point (P) on its axis is $2\pi(1\text{-}\cos\theta)$, where θ is the angle subtended by any radius at P. If r is the distance of the periphery from P, x is the distance of P from the centre and a is the radius of the circle, then

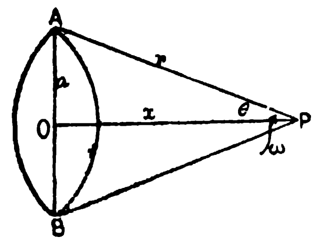


Fig. 2·6

$$r^2=a^2+x^2$$

Hence $\omega=2\pi\left[1-\dfrac{x}{\sqrt{a^2+x^2}}\right]$

So $V=i\omega=2\pi i\left[1-x(a^2+x^2)^{-\frac{1}{2}}\right]$

Since $H=-\dfrac{dv}{dx}$, $H=\dfrac{2\pi i a^2}{(a^2+x^2)^{\frac{3}{2}}}$

Field at the centre of the coil, where $x=a$, is $H=\dfrac{2\pi i}{a}$

BY APPLICATION OF LAPLACE'S LAW : Consider an element δs of a circular coil at A, the coil being regarded as

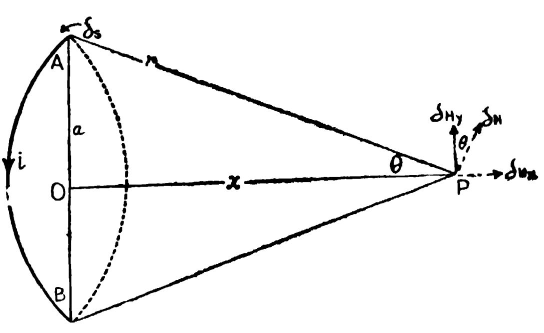


Fig. 2.7

placed at right angles to the plane of the paper. The intensity of the magnetic field at P due to this element acts at right

angles to AP and is in the plane of the paper. Its magnitude is given by

$$\delta H = \frac{i.\delta s}{r^2}$$

Since the angle between δs and r is $\frac{\pi}{2}$, δH makes an angle $\left(\frac{\pi}{2} - \theta\right)$ with the positive direction of OP. Resolving δH into two rectangular components δH_x and δH_y along and at right angles to OP, we get,

$$\delta H_x = \frac{i.\delta s.\ \sin\theta}{r^2} \quad \text{and} \quad \delta H_y = \frac{i.\delta s.\ \cos\theta}{r^2}$$

By considering all such symmetrical pairs of elements into which the circular conductor may be divided, we get

$$H_x = \Sigma\,\delta H_x = \frac{i.\ \sin\theta}{r^2}\Sigma\,\delta s = \frac{2\pi ai\ \sin\theta}{r^2} = \frac{2\pi a^2 i}{r^3}$$

and $H_y = \Sigma\,\delta H_y = 0$, because of the fact that the components at the ends of a diameter are equal but mutually opposite.

Thus the field at any point on the axis of a circular coil is directed along the axis and is given by

$$H = \frac{2\pi i a^2}{r^3} = \frac{2\pi i a^2}{(a^2+x^2)^{\frac{3}{2}}}$$

Field at the centre of the coil, $H = \dfrac{2\pi i}{a}$

Field inside a Solenoid : Let AB (fig. 2·8) be the axis of a

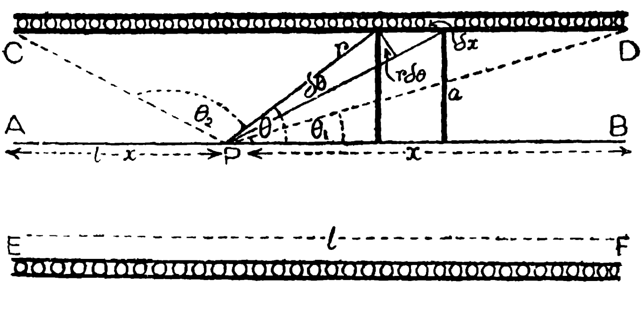


Fig. 2·8

short solenoid of length l, radius a, having n turns per unit length. Let i be the current (*e.m.u.*) flowing in each turn.

Consider a small element δx at a distance r from a point on the axis. This element may be regarded as a circular coil consisting of $n.\delta x$ number of turns and field due to this at P is given by

$$H=\frac{2\pi na^2 i\delta x}{r^3}$$

As shown in the sketch,

$$r\delta\theta=\delta x \sin\theta$$

$$\text{So} \quad H=\frac{2\pi na^2 ir.\delta\theta}{r^3 \sin\theta.}$$

$$\text{Since } \frac{a}{r}=\sin\theta, \quad H=2\pi ni \sin\theta.\delta\theta$$

For the whole solenoid, if θ_1 and θ_2 be the angles subtended at P by the extremities,

$$H=2\pi ni\int_{\theta_1}^{\theta_2}\sin\theta.d\theta=2\pi ni\,[\cos\theta_1-\cos\theta_2]$$

If the solenoid is long and of small diameter amounting to an *infinite solenoid*, so that $\theta_1=0$ and $\theta_2=\pi$, the field at any point inside the solenoid is given by $H=4\pi ni$.

Further, by expressing $\cos\theta_1$ and $\cos\theta_2$ in terms of the distances of P from the two ends, written as x and $(l-x)$

$$H=2\pi ni\left[\frac{x}{\sqrt{x^2+a^2}}+\frac{l-x}{\sqrt{(l-x)^2+a^2}}\right]$$

At the centre of the solenoid $x=\frac{l}{2}$,

$$\text{so field at the centre, } H=2\pi ni\,\frac{2l}{\sqrt{l^2+4a^2}}=4\pi ni\left[1-\frac{2a^2}{l^2}\right]$$

$$\text{If } l\rightarrow\infty,\ H=4\pi ni.$$

Ampere-turns : When the current through a solenoid is expressed in amperes, the product of the total number of turns and current (*i.e.* nli) is called the ampere-turns of the solenoid.

Magnetic Potential on the axis of a Solenoid : The solid angle subtended by a small circular element δx at a point P on the axis is $2\pi(1-\cos\,\theta)$, where θ is the angle subtended by any radius at P. If the coil carries a current i (*e.m.u*) the potential due to it at P is

$$\delta V = 2\pi ni.\delta x(1-\cos\,\theta)$$

$$= 2\pi ni\,\frac{i\delta\theta}{\sin\theta}\,(1-\cos\theta), \quad \text{since } r.\delta\theta=\delta x.\sin\,\theta$$

or $$\delta V = 2\pi nia\left(\frac{1-\cos\,\theta.\delta\theta}{\sin^2\theta}\right), \text{ since } a=r\sin\,\theta$$

The potential due to whole solenoid is given by

$$V = 2\pi ni\,a\int_{\theta_1}^{\theta_2}\frac{1-\cos\,\theta}{\sin^2\theta}\,d\theta.$$

or $$V = 2\pi nia\Big[-\cot\,\theta+\text{cosec}\,\theta\Big]_{\theta_1}^{\theta_2}$$

or $$V = 2\pi nia\left[\frac{1-\cos\,\theta_2}{\sin\,\theta_2}-\frac{1-\cos\,\theta_1}{\sin\,\theta_1}\right]$$

Magnetic field due to a toroid : If a solenoid is bent round so that the two ends meet and its axis forms a circle of radius r, we have an endless solenoid in the form of an *anchor ring* also called *Rowland ring* or a toroid. Let H be the field at any point on a circular path inside of length equal to $2\pi r$ and having its centre on the linear axis of the ring. The line integral of the field along this circular path is $2\pi rH$. Hence applying Ampere's Circuital theorem, we may write

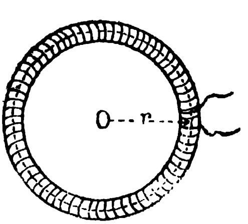


Fig. 2·9

$$2\pi rH = 4\pi(2\pi rni)$$

or $$H = 4\pi ni$$

The field external to the toroid, which however arises due to leakage of magnetic flux is negligibly small.

ILLUSTRATIVE EXAMPLES

1. *A conductor in the form of a regular hexagon of side $4\sqrt{3}$ cm. carries a current of 2 amperes. Calculate the field at the centre of the hexagon.*

Solution : Field due to a straight conductor is given by the equation

$$H = \frac{i}{r}(\cos\alpha + \cos\beta)$$

Let us first obtain the field at the centre to the conductor AB.

$$AB = AO = 4\sqrt{3}\ cm.$$

$$\alpha = \beta = 60^\circ$$

$$OP = AP \tan 60^\circ = 6\ cm.$$

Hence $H = \frac{0 \cdot 2}{6}\left(2 \times \frac{1}{2}\right) = \frac{1}{30}$

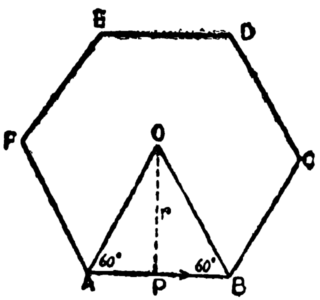


Fig. 2 10

The field is directed perpendicular to the plane of the paper and acts upwards if the current is anti-clockwise round the hexagon. The magnitude and direction due to other sides are same as this. Hence the resultant field is $\frac{6}{30} = 0 \cdot 2$ oersted.

2. *A flat circular coil having 100 turns of mean radius* 16 *cm. is set with its plane vertical and making an angle 30° with the magnetic meridian. Calculate the current that must be passed through the coil so that a magnet suspended at its centre is urged to set it at right angles to magnetic meridian.* ($H = 0 \cdot 37$ *oersted*)

Solution : As shown in the sketch (fig. 2·11) the field H due to the coil makes an angle 60° with the earth's field (H_0). If the magnet be in equilibrium in a position perpendicular to the magnetic meridian, then $H_0 = H \cos 60^\circ$.

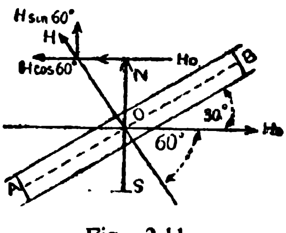


Fig. 2 11

$$H = \frac{2\pi ni}{a}, \text{ So } \frac{2\pi ni \cos 60^\circ}{a} = H_0$$

or $$i = \frac{aH_0}{2\pi n \cos 60^\circ} = \frac{16 \times 0 \cdot 37 \times 2}{2\pi \times 100}$$

or $i = 0 \cdot 88$ ampere.

3. *Two solenoids each of radius 8 cm. and length 9 cm. having 34 turns per centimetre are placed co axially with a gap of 12 cm. between them. Calculate the field at the centre*

of the gap when a current of 2·5 amperes flows through the two solenoids in series producing a cumulative effect.

Solution : The field due to a solenoid is given by

$$H=2\pi ni\,[\cos\theta_1-\cos\theta_2]$$

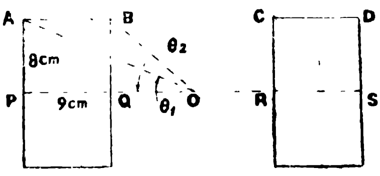


Fig. 2·12

The two solenoids produce equal fields in the same direction. Hence the resultant field is twice that due to a single solenoid. So the field is obtained as shown.

$$\cos\theta_1=\frac{OP}{OA}=\frac{15}{\sqrt{8^2+15^2}}=\frac{15}{17},\quad \cos\theta_2=\frac{6}{\sqrt{8^2+6^2}}=\frac{6}{10}$$

$$\cos\theta_1-\cos\theta_2=\frac{15}{17}-\frac{6}{10}=\frac{48}{170}$$

Therefore $H=2\times2\pi\times34\times0{\cdot}25\times\frac{48}{170}=30{\cdot}87$ oersteds.

III-3. ACTION OF MAGNET ON CURRENT

Force on a Conductor in a Magnetic Field : Laplace's law expressed symbolically as $\delta H=\dfrac{i.\,\delta s.\,\sin\theta}{r^2}$ gives the magnetic field produced by a short segment of a conductor. The force exerted on a pole m by such an element is given by

$$\delta F=m.\delta H=\frac{m.\,i.\,\delta s.\,\sin\theta}{r^2}$$

By Newton's third law of motion the element would experience an equal and opposite force for being placed in a field due to the pole. It may be considered that $\dfrac{m}{r^2}$ is the magnetic induction (B) due to the pole at the position of element of the conductor. Hence the force on a conductor of length δs in a field of induction B is given by

$$\delta F=B.i.\sin\theta.\delta s$$

$$\text{or}\quad \delta\mathbf{F}=i[\mathbf{B}\times\delta\mathbf{s}]$$

$\delta\mathbf{F}$ acts at right angles to the plane containing δs and $\mathbf{B}$.

This expression may be deduced independently.

Consider a circuit ABC in which a current i is flowing, placed at a distance r from a pole of strength m kept at a fixed point P.

Let the circuit be displaced under the action of the magnetic field always remaining parallel to itself, through a distance δx and occupy a new position $A'B'C'$. Work is done by the displacement and this causes an equal amount of change of potential energy of the system comprised of the pole and the conductor.

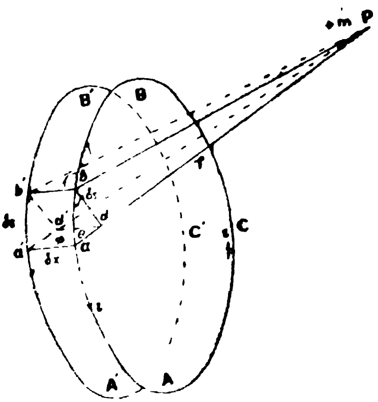


Fig 2·13

If F is the force acting on unit length of the conductor, $F.\delta s$ is the force on an element ab of length δs acting in the direction of the displacement, which makes an angle ϕ with ab. The work done on the element is $F.\delta s.\delta x$ and for the whole circuit the work is given by

$$W = \Sigma\ F.\delta s.\delta x$$

Again, the change in potential of the circuit due to changed position is calculated as (current $\times$ change in solid angle). The area swept out by δs is $\delta s.\delta x.\sin\phi$. The area normal to the direction of P is $\delta s.\delta x.\sin\phi.\ \sin\theta$, where θ is the angle made by δs with the direction of P.

Change in solid angle due to the shift of δs is

$$\frac{\text{normal area swept out by } \delta s}{r^2} = \frac{\delta s.\delta x.\ \sin\phi.\ \sin\theta}{r^2}$$

Change in the solid angle considering the whole surface is

$$\Sigma\frac{\delta s.\ \delta x.\ \sin\theta.\ \sin\phi}{r^2}$$

Change in potential $= i.\Sigma\dfrac{\delta s.\ \delta x.\ \sin\theta.\ \sin\phi}{r^2}$

Since the pole is of strength m, the change in potential energy of the system is given by

$$W=mi\ \Sigma\ \frac{\delta s.\ \delta x.\ \sin\phi.\ \sin\theta}{\ }$$

Equating this to the work done by the force, we get

$$\Sigma\ F.\delta x=\frac{mi}{r^2}\ \Sigma\ \delta s.\ \delta x.\ \sin\phi.\ \sin\theta.$$

or, $\quad F=Bi.\delta s.\ \sin\phi.\ \sin\theta$

B is the induction in the region of ABC due to pole m.

The force becomes maximum for a given value of θ if $\phi=90°$. Since the change in potential energy in any system always tends to be maximum, the effective force will be $Bi \sin\theta$ per unit length and at right angles to ab. For a similar reasoning the angle made by the direction of displacement with the direction of the magnetic field *i.e.* with r is also 90°, since for a given displacement in any direction the work done is greatest when the solid angle subtended by the circuit is changed most for that displacement. Hence the direction of the resultant force is at right angles to the plane containing the element of the current and the direction of the magnetic field.

Alternative method : Potential energy of a magnetic shell in a magnetic field is given by (strength of shell $\times$ total flux through it). For a closed circuit carrying a current i, the strength of the equivalent shell is i and so the potential energy is $N.i$, where N is the total flux through the circuit due to the field in which it is placed.

Let an area of surface enclosed by a conductor carrying a current i be s. Let it be displaced by a force F acting on it in the field and let S' be the area of the field occupied by it in the new position. Suppose N and $N+\delta N$ be the flux through the surface in its two positions respectively. So the decrease in potential energy is

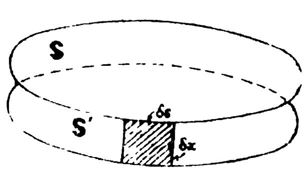


Fig. 2·14

$$i.(N+\delta N)-i.N=i.\delta N$$

Consider that a small length δs of the conductor is displaced through δx by the force due to the field sweeping out an area

$\delta\mathbf{s}\times\delta\mathbf{x}$. Imagine the volume of space enclosed between the two areas S and S' and bounded by the surface $\Sigma\delta\mathbf{s}\times\delta\mathbf{x}$. The total normal induction over the surface enclosing the volume consists of three parts, the induction through s, s′ and $\Sigma\,\delta\mathbf{s}\times\delta\mathbf{x}$. The first two are as stated before N and $N+\delta N$. To find the third we may proceed as follows.

If B is the magnetic induction at δs, the flux through area $\delta\mathbf{s}\times\delta\mathbf{x}$ is $\mathbf{B}.\delta\mathbf{s}\times\delta\mathbf{x}$. Considering the rules involving triple scalar product this may be written as $\mathbf{B}\times\delta\mathbf{s}.\delta\mathbf{x}$. Hence the flux of magnetic induction across the surfaces swept out by the shell is given by

$$\int\int\mathbf{B}\times d\mathbf{s}.d\mathbf{x}=\int_0\mathbf{B}\times d\mathbf{s}.\int d\mathbf{x}=\delta\mathbf{x}.\int_0\mathbf{B}\times d\mathbf{s}$$

$\int_0$ indicates integration round the periphery of the shell.

Therefore the total normal induction over the volume under consideration is

$$(N+\delta N)-N+\delta\mathbf{x}.\int\mathbf{B}\times d\mathbf{s}$$

This is zero by Gauss's theorem. Hence

$$\delta N=-\delta\mathbf{x}\int_0\mathbf{B}\times d\mathbf{s}$$

As stated before, the change in potential energy is $i.\delta N$ and substituting for δN, this change may be written as

$$-\,i.\,\delta\mathbf{x}\int_0\mathbf{B}\times d\mathbf{s}$$

Equating this with the work done by the total force $\mathbf{F}$ acting on the shell

$$\mathbf{F}.\,\delta x\ =\ -i.\,\delta\mathbf{x}.\int_0\mathbf{B}\times d\mathbf{s}\ =\ i.\,\delta\mathbf{x}.\int_0 d\mathbf{s}\times\mathbf{B}$$

$$\text{Hence}\quad \mathbf{F}=i\int_0 d\mathbf{s}\times\mathbf{B}$$

This expression suggests that the total force $\mathbf{F}$ on the circuit

is the same as if the force on each element δs is expressed as $\delta \mathbf{F} = i[\delta s \times \mathbf{B}]$. $\delta \mathbf{F}$ is normal to the plane containing δs and **B**.

FLEMING'S LEFT HAND RULE : The quantities F_1,B and i are related to one another by a rule due to Fleming. If the thumb, forefinger and middle finger of the left hand are extended so that they are mutually at right angles and the mIddle finger is pointed to the direction of i (current), the Forefinger to the direction of field, the thuMb would indicate the direction of motion when the circuit moves due to the action of the field.

Couple on a coil in a field : Let a rectangular coil *abcd* carrying a current i be placed in a field H, the normal to the plane of the coil making an angle θ with the field. The force on each of the sides *ad, bc* due to the magnetic field is of magnitude Hil, where l is the length of *ad* or *bc*. The forces on the two sides are equal and parallel but mutually opposite and so they form a couple tending to rotate the coil so as to bring the normal of the coil in the same line as the field. The forces on the sides *ab* and *cd* are vertical and mutually opposite and as such they do not constitute a couple and contribute nothing towards rotation or motion.

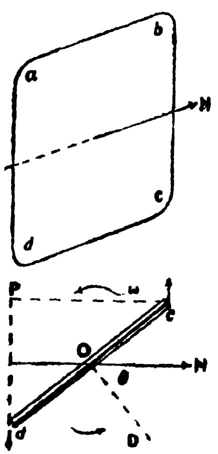


Fig. 2·15

The moment of the couple is given by $C = Hil\ w \sin\theta$, where w is the width of the coil.

Since $l.w = A$, the area of the coil, hence

$$C = HiA \sin\theta.$$

The couple varies with the position of the normal to the plane of the coil with respect to the field. The couple is maximum when $\theta = 90°$. This happens when the plane of the coil is parallel to the field. If the field be radial one, the plane of the coil in all its positions remain parallel to the field and in such a case the couple is of constant magnitude HAi.

If there be n similar turns in the coil, the effective couple is $nHAi$.

The same result is obtained by considering the equivalent shell of the coil. The couple acting on a magnet of moment M when placed with its axis at an angle θ with the field H is $MH \sin\theta$. The magnetic moment of the shell equivalent to a coil of area A carrying a current i is Ai. So when the coil is in a field H, its normal (the axis of the equivalent shell) making an angle θ with H, the couple acting on it is $HAi \sin\theta$. In a radial field this is HAi. It may be noted that the couple as calculated is irrespective of the shape of the coil.

III-4. ACTION OF CURRENT ON CURRENT

Force between two straight parallel wires : A linear current i_1 produces a field $H=2i_1/r$ at a distance r from it in a direction perpendicular to its length. A second linear conductor carrying a current i_2, parallel to the first experiences a force $Hi_2=2i_1i_2/r$ per unit length. There is mutual attraction between them when the currents in the two are in the same direction and mutually opposite currents cause a repulsion.

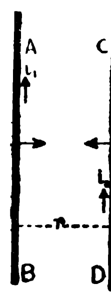


Fig. 2·16

Two circular coils mutually at right angles : Let AB be the larger coil of the two (fig. 2·17) having a radius a and carrying a current i_1. The other coil CD is small in size and is in the central portion of AB. If α is the area of CD and i_2 the current in it, its magnetic moment is αi_2.

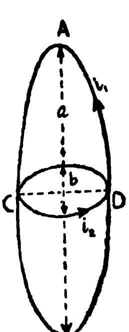


Fig 2·17

The field due to AB at its centre is $\dfrac{2\pi i_1}{a}$.

CD experiences a couple $\dfrac{2\pi i_1}{a}.\alpha i_2$ tending to rotate it so as to bring it in the same plane as AB. When the coils are initially inclined to each other at an angle θ, the couple is $\dfrac{2\pi i_1 i_2 \sin\theta}{a}$.

Co-axial coils : Let two circular coils, of very nearly the

same radius and carrying currents i_1 and i_2 respectively, be placed at a small distance x apart.

The force on unit length of either, say A and C respectively, considering a small element as a straight conductor is $\frac{2i_1i_2}{AC}$ in the direction of AC. The components of this normal to the axis taken all round the periphery of the coils mutually cancel and the components parallel to the axis add up and may be obtained as

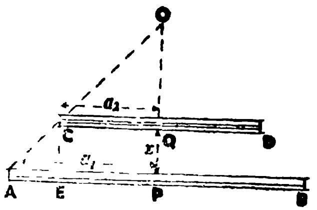


Fig. 2·18

$$\text{Force}=\frac{2i_1i_2}{AC}\cdot\frac{CE}{AC}=\frac{2i_1i_2.x}{AC^2} \text{ for unit length.}$$

Considering the whole coil since the total length is $2\pi a_2$ and $AC^2=AE^2+CE^2=(a_1-a_2)^2+x^2$,

$$\text{Force}=\frac{2i_1i_2.x.2\pi a^2}{(a_1-a_2)^2+x^2}=\frac{4\pi i_1i_2a_2x}{r^2+x^2}$$

This becomes zero when $x=0$. *i.e.*, when the two coils are in the same plane. This is maximum when $\frac{x}{r^2+x^2}$ is greatest. This occurs if

$$\frac{d}{dx}\left(\frac{x}{r^2+x^2}\right)=\frac{r^2+x^2-2x^2}{(r^2+x^2)^2}=\frac{r^2-x^2}{(r^2+x^2)^2}=0$$

That is when $x=r=a_1-a_2$,

The maximum force is given by $\frac{2\pi i_1i_2a_2}{a_1-a_2}$.

If one of the coils is very small (fig. 2·19), this may be regarded as a magnetic shell of thickness δx and pole strength m per unit area.

Force on its under face $=H.m.\pi a_2^2$

Force on the upper face$=\left(H+\frac{dH}{dn}\delta x\right)m.\pi a_2^2$

The resultant force is the difference of these two forces and so it is given by $\frac{dH}{dx}.m.\delta x.\pi a_2^2$. Now $m.\delta x$ is the magnetic moment per unit area of the equivalent shell and so it is equal to i_2. Therefore the resultant force is given by

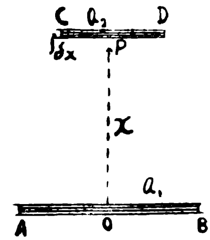


Fig. 2·19

$$\text{Force} = \pi a_2^2\, i_2 . \frac{dH}{dx}$$

$$\text{Now } H = \frac{2\pi a_1^2 i_1}{(a_1^2+x^2)^{\frac{3}{2}}}$$

$$\text{So } \frac{dH}{dx} = -\frac{6\pi^2 a_1^2 i_1 x}{(a_1^2+x^2)^{\frac{5}{2}}}$$

$$\text{Therefore Force} = \frac{6\pi a_1^2 a_2^2 i_1 i_2 . x}{(a_1^2+x^2)^{\frac{5}{2}}}$$

When $x=0$, the force vanishes. It is maximum when $x=\frac{a_1}{2}$.

II-5. CURRENT MEASURING INSTRUMENTS

General Principles : Magnetic effect provides for convenient methods of measuring electric current. Magnetic field produced by a coil is proportional to the intensity of current flowing through it and again the magnetic effect may be applied to produce motion in a magnet. Conversely, a magnet may be used to cause force effect in a coil carrying current. Such contrivances form the basic principles of current measuring instruments, known as *Galvanometers*. Mutual action between current elements is utilised in *dynamometer* type of instruments.

The principle underlying all electromagnetic measuring instruments is that the current carrying circuit tends to enclose maximum flux when free to move in a magnetic field. In a moving magnet instrument the magnet turns so that more of its flux passes through the coil. In a moving coil instrument the coil moves so as to enclose as much as possible of the

flux due to the magnet. In a dynamometer the moving coil turns so that its magnetic field adds up to that of the fixed coil.

Tangent Galvanometer : The magnetic field at the centre of a circular coil carrying current i (*e.m.u.*) having n turns each of radius a is given by $H=2\pi ni/a$. This equation is used in a Tangent galvanometer to measure current.

At the centre of a circular coil there is a suspended or pivoted magnetic needle free to rotate in a horizontal plane about a vertical axis through its centre of gravity. The magnet is of short length. This is necessary to obtain that the needle moves in a uniform field at the central region.

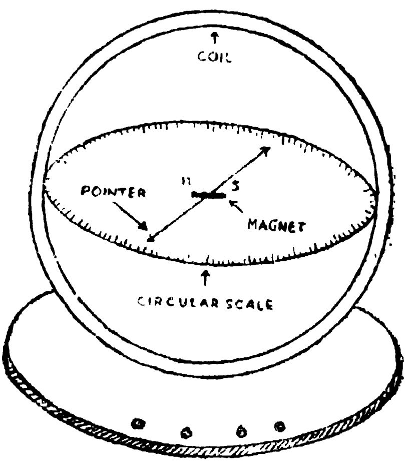


Fig. 2·20

The coil is placed in the magnetic meridian. When a steady current passes through the coil, a magnetic field (H) is produced at right angles to the earth's horizontal field (H_0). The needle is as such subjected to two fields, which are mutually perpendicular. If θ is the deflection of the magnet from magnetic meridian, then by tangent law

$$H=H_0 \tan\theta$$

$$\text{or} \quad \frac{2\pi ni}{a}=H_0 \tan\theta$$

$$\text{or} \quad i = \frac{aH_0}{2\pi n}\tan\theta = \frac{H_0}{G}\tan\theta.$$

G stands for $2\pi n/a$, called the galvanometer constant. Writing $H_0/G=K$, $i=K\tan\theta$. This constant K is called the *reduction factor*. If the current is to be obtained in amperes, i should be equated to $10K\tan\theta$.

The deflection is observed by means of a pointer attached at right angles to the needle and moving over a circular scale graduated in degrees. If the constants involved in H_0/G are obtained, the instrument may be used for absolute measurement of current in e.m. units. But it is difficult to measure the constants with precision.

Sensitivity : If a small change δi of current produces a change $\delta\theta$ in deflection, then $\frac{\delta\theta}{\delta i}$ is a measure of sensitivity and for better working this should be large. In the limit

$$\frac{\delta\theta}{\delta i}=\frac{d\theta}{di}.$$

$$i=K\tan\theta$$

$$di=K\sec^2\theta.d\theta$$

$$\text{or}\quad \frac{d\theta}{di}=\frac{\cos^2\theta}{K}$$

$\frac{d\theta}{di}$ tends to be maximum as $\cos\theta\to 1$. This means that the instrument shows better sensitivity when the deflection is small.

Greatest Accuracy : If $\delta\theta$ be an error in reading the deflection, let the corresponding error in the calculated value of the current be δi. Then $\frac{\delta i}{i}$ is the fractional error and so $\frac{\delta\theta}{\delta i/i}$ is a measure of accuracy. For good working $\frac{\delta i}{i}$ should be small and so $\frac{i.\delta\theta}{\delta i}$ should be maximum. In the limit $\frac{\delta\theta}{\delta i}=\frac{d\theta}{di}$ and so

$$i\frac{d\theta}{di}=K\tan\theta.\frac{\cos^2\theta}{K}=\tfrac{1}{2}\sin 2\theta$$

Accuracy will be greatest when $\sin 2\theta=1$, *i.e.* when $\theta=\frac{\pi}{4}$. Hence for accuracy it is desirable to keep the deflection as near as possible to 45°.

HELMHOLTZ DOUBLE COIL GALVANOMETER : In the ordinary tangent galvanometer as described, the magnetic

needle however short does not move in uniform field. Further, the field at the centre of the coil may not be same as the field in the region where the poles of the magnet are placed in any deflected position. To remove this difficulty Helmholtz designed a type of tangent galvanometer in which two circular coils are so placed that their axis coincide and the centres are separated by a distance equal to their common radius. The magnet is suspended at a point midway between the coils on the common axis, where, as shown below, the field is practically uniform.

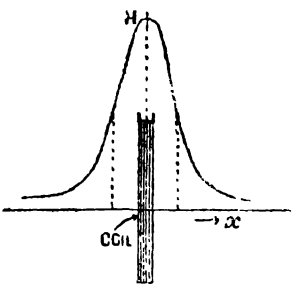


Fig. 2·21

The variation of intensity of field along the axis of a circular coil may be represented by the curve shown (Fig 2·21). It is observed that a small portion of the curve is straight which means that at some distance from the centre of the coil $\frac{dH}{dx}$ is constant. This distance is obtained as shown below.

We have,

$$H=\frac{2\pi na^2i}{(a^2+x^2)^{\frac{3}{2}}}=K\,(a^2+x^2)^{-\frac{3}{2}}\text{, writing } K \text{ for } 2\pi na^2i.$$

$$\text{So } \frac{dH}{dx}=-K.3x\,(a^2+x^2)^{-\frac{5}{2}}$$

$$\text{If } \frac{dH}{dx}=\text{constant, } \frac{d^2H}{dx^2}=0$$

$$\text{So } \frac{d^2H}{dx^2}=-3K\left[(a^2+x^2)^{\frac{5}{2}}-\frac{5x}{2}(a^2+x^2)^{-\frac{7}{2}}.2x\right]=0$$

$$\text{Hence we get } x=\pm\frac{a}{2}.$$

In the arrangement made with two coils, as stated, the field midway between the coils should therefore be uniform, as the increase of field due to one is compensated by an equal decrease due to the other. The coils are of same dimensions having the same number of turns and these are connected in series in such

a way that they produce the field in the same direction. The

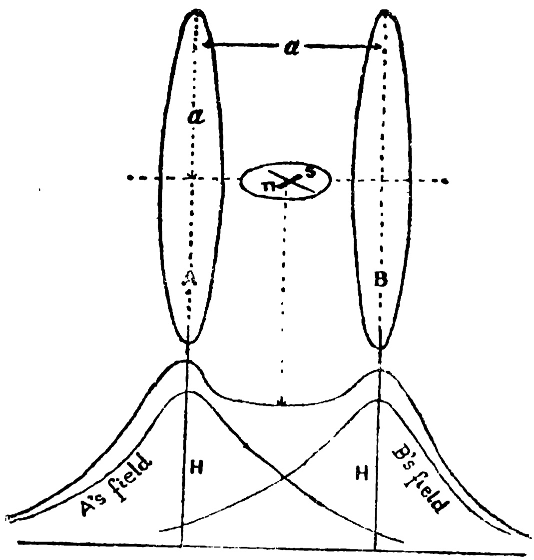


Fig. 2·22

field due to a coil of n turns each of radius a, carrying a current i, at a point on the axis at a distance x from the centre is

$$H=\frac{2\pi na^2i}{(a^2+x^2)^{\frac{3}{2}}}$$

If there are two coils producing field in the same direction and if $x=\frac{a}{2}$,

$$H=\frac{32\pi ni}{5\sqrt{5}a}$$

For equilibrium in mutually perpendicular fields (one of them being H_0, due to earth's horizontal intensity), if θ be the shift of magnet from the direction of H_0, we have

$$\frac{32\pi ni}{5\sqrt{5}a}=H_0\tan\theta$$

$$\text{or } i=\frac{5\sqrt{5}}{32}\cdot\frac{aH_0}{\pi n}\tan\theta=K\tan\theta$$

$$\text{where } K=\frac{5\sqrt{5}}{32}\cdot\frac{aH_0}{\pi n}.$$

Suspended coil galvanometer : By Ampere's theorem a closed coil of n turns each of area A (effective area is nA) when carrying a current i (e.m.u.) has a magnetic moment $M = nAi$.

When placed in uniform field of intensity H with its axis (*i.e.* normal to the plane of the coil) inclined at an angle ϕ with the field, the coil experiences a couple $nAiH \sin \phi$ tending to bring the axis in a line parallel to the field. If however $\phi = 90°$, the couple reduces to $nAiH$.

A rectangular coil containing a good number of turns is suspended with a phosphor bronze strip (P) within the cylindrical pole pieces of a U-shaped permanent magnet (Fig 2·23). When a current flows through the coil a couple acts on it and causes it to rotate until the twist in the suspension produces an equal and opposite torque. To make the deflecting couple as strong as possible, the number of turns in the coil is increased and the field is made strong. Moreover, in order that the plane of the coil may in any deflected position remain parallel to the field, the field is made radical (fig 2·23b) by inserting a cylindrical piece of soft iron inside the coil midway between the pole pieces. The current enters the coil through the suspension strip and leaves through a fine spring below the coil. This spiral spring also exerts a small restoring couple on the coil.

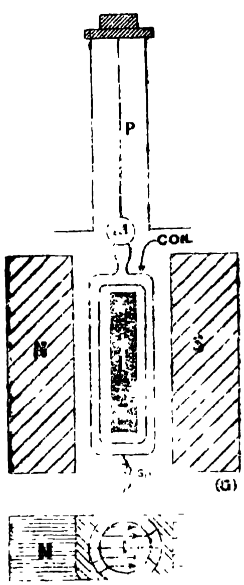


Fig. 2·23

Let θ be the steady angular deflection and c be the couple in the suspension strip due to unit twist (in radian). In equilibrium position the restoring couple $c\theta$ is equal to the restoring couple, so

$$nAHi = c\theta$$

$$\text{or} \quad i = \frac{c}{nAH}\theta$$

$$\text{so} \quad i \propto \theta$$

That is the deflection of the coil is a measure of current. Deflection per unit current is θ/i, which determines the sensitivity of the galvanometer. The expression $\frac{\theta}{i}=\frac{nAH}{c}$ shows that the sensitivity increases for greater number of turns, stronger field and larger area of the coil.

The coil in this case is totally enclosed in a narrow gap where there is a strong magnetic field and so this type of galvanometer is unaffected by earth's magnetic field or any stray magnetic field outside. The rotation of the coil is observed as the shift of a spot of light reflected from a piece of mirror (M) placed on the suspension strip. Let the spot of light shift through a distance d on a scale placed at a distance D from the mirror. When the mirror turns through an angle θ, then as shown in sketch (Fig. 2·24),

$$\tan\ 2\theta=\frac{d}{D}$$

If θ is small, $$2\theta=\frac{d}{D}$$

i.e. $$\theta=\frac{d}{2D}$$

Therefore $$i=K\theta=K\frac{d}{2D}$$

So $i \propto d$.

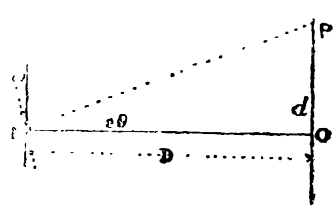


Fig. 2·24

This type of galvanometer is better known as *d'Arsonval galvanometer*. The coil instead of being rectangular may be circular as well and in such a case the soft iron piece inside the coil is taken in a spherical form.

The *figure of merit* of a mirror galvanometer is the current in amperes that produces a deflection of one millimetre of a spot of light reflected from the mirror of the galvanometer on a scale at a distance of one metre from the mirror. The smaller the number necessary for expressing the figure of merit, the more sensitive is the galvanometer.

The *current sensitivity* of a mirror galvanometer is defined as the deflection in millimetre produced on a scale one metre away from the galvanometer mirror of a spot of light reflected

from the mirror by one micro-ampere current. A higher value of current sensitivity relatively indicates better sensitiveness. Current sensitivity is numerically the reciprocal of figure of merit expressed in micro-amperes.

The *voltage sensitivity* is the deflection in millimetres to be obtained on a scale one metre away from the mirror of a spot of light reflected from the mirror when a voltage of one micro-volt is applied at the galvanometer terminals. The higher the number expressing the voltage sensitivity, the more sensitive is the galvanometer.

If d is the voltage sensitivity, it is the deflection produced by one micro-volt applied at the terminals. If R is the resistance of the galvanometer coil in ohms, the current in such a case is $1/R$ micro-ampere. So 1 micro-ampere current through the coil would produce a deflection $R.d$ millimetres. Hence

Current sensitivity = Voltage sensitivity × galv. resistance.

Some authors define current sensitivity in same terms as figure of merit and they consider voltage sensitivity as the voltage required to obtain a deflection of one millimetre. According to these definitions,

Voltage sensitivity = current sensitivity (fig. of merit) *× galv. resistance.*

A voltage sensitive galvanometer used for null point determinations should have a low resistance coil.

Factors controlling current sensitivity : The rotation of the coil by an angle θ causes the shift of the spot of light by 2θ and if d is the deflection in millimetres of the spot on the scale at one metre distance from the mirror, then

$$2\theta = \frac{d/10}{100} \text{ radian}$$

$$\text{or} \quad \theta = \frac{d}{2000} \text{ radian}$$

$$\text{So} \quad i = \frac{c\theta}{nAH} = \frac{c \times d}{2000nAH} \text{ e.m. units}$$

$$\text{or} \quad d = \frac{2000nAHi}{c} \text{ mm.}$$

Shift of spot in millimetres produced by 10^{-6} amp. *i.e.* 10^{-7} e.m. unit of current is

$$x=\frac{2000nAH\times10^{-7}}{c}=\frac{2nAH\times10^{-4}}{c}$$

So current sensitivity is $\dfrac{2nAH\times10^{-4}}{c}$ mm. per micro-ampere.

DEAD BEAT GALVANOMETER : The moving coil of a galvanometer is a suspended system and it is likely to oscillate when once displaced from its position of equilibrium. So when a current is withdrawn, the deflected coil would oscillate for a long time before coming to rest. In order to avoid this the system is so designed that its oscillations are quickly damped. This is obtained by winding the coil on a metallic frame. In such a case in course of oscillations in a magnetic field the frame forming by itself a closed circuit generates induced current in it. According to Lenz's law this induced current opposes the oscillations. As such oscillations are soon stopped. This is known as *electro-magnetic damping.* Further, by short circuiting the oscillating coil (when the deflecting current has been withdrawn) by a shunt key closed across the galvanometer terminals, a stronger *emf* is caused to be induced in the oscillating coil itself causing electro-magnetic damping all the more effective.

REQUIREMENTS OF A GOOD GALVANOMETER : A galvanometer intended to work satisfactorily should (i) have a steady zero-position, (ii) have very little effect due to temperature variation, (iii) have a robust construction and good insulation and (iv) be free from disturbing effects of dust and air currents.

Steady zero is obtained better in 'taut suspension' where the coil instead of being suspended is stretched between two wires. Air-tight case protects the instrument from dust and air current.

In portable type galvanometers the coil has either taut suspension or pivot movement. The control in a pivoted coil is due to hair springs. A pointer attached to the coil moves on a calibrated dial.

II-6. CHARGE MEASURING INSTRUMENTS

Ballistic galvanometer : This type of galvanometer is used to measure the quantity of charge circulating a closed circuit in a short interval of time. Due to flow of charge the coil receives an impulse and the throw is a measure of it. It is therefore essential that the whole of the charge circulates the coil before it starts moving. To achieve this a coil of high moment of inertia and slow motion is necessary. Such a galvanometer may be constructed either as a moving coil or as a moving magnet instrument. Oscillations are set up in a system subject to a restoring force proportional to displacement when it is given an impulse while at rest. But if there are dissipative forces, the oscillations are damped. To measure the impulse correctly the dissipative forces are to be minimised.

MOVING COIL INSTRUMENT : In construction it resembles a suspended coil galvanometer described in the previous section (Fig. 2·23).

Let H be the strength of the magnetic field in which a suspended coil of vertical length l and breadth b is placed. If the coil carries a current i, the force on each vertical side of it is Hil. If the current lasts for a time t, the impulse from start to finish is given by

$$\int_0^t Hli.dt = Hl\int^t i.\,dt = Hl\int^q dq = Hlq$$

q is the total charge circulated through the galvanometer in time t.

The moment of impulse about the axis of suspension considering that there are two sides, is $2Hl.\frac{1}{2}bq = Hlbq = HAq$, where A is the area $(l \times b)$ of the coil. If there are n turns each of same area, the effective moment or the torque applied is $nAHq$.

If I be the moment of inertia of the coil and ω its angular velocity, the angular momentum is $I\omega$. Hence according to second law of motion applied to angular velocity, we may write

$$I\omega = naHq$$

$$\text{or} \quad \omega = \frac{nAHq}{I} \qquad \text{..........} \quad \text{(i)}$$

The system starts with a velocity ω and its initial kinetic energy is $\frac{1}{2} I\omega^2$. This energy is completely used up in doing work in twisting the suspension through an angle θ, the maximum and undamped throw. If c is the restoring couple for unit twist, $c\alpha$ is the couple in a deflected position α. The work done for an additional twist $\delta\alpha$ is $c\alpha.\,\delta\alpha$. Hence the total work done when the deflection reaches θ is

$$W=\int_0^\theta c\alpha.d\alpha=\tfrac{1}{2}\,c\theta^2$$

Hence, $\frac{1}{2}\,I\omega^2=\frac{1}{2}c\theta^2$

$$\text{or}\quad \omega^2=\frac{c}{I}\theta^2 \qquad \dots\ (\text{ii})$$

From (i) and (ii), $\left[\frac{nAHq}{I}\right]=\frac{c}{I}\theta^2$

$$\text{or}\quad q^2=\frac{c^2}{n^2A^2H^2}.\frac{I}{c}\theta^2 \qquad \dots\dots\dots\dots(\text{iii})$$

Again, if T is the period of oscillation of the suspended coil, then

$$T=2\pi\sqrt{\frac{I}{c}}$$

$$\text{or}\quad \frac{I}{c}=\frac{T^2}{4\pi^2} \qquad \dots \qquad \dots\ (\text{iv})$$

From (iii) and (iv) we get,

$$q^2=\frac{T^2}{4\pi^2}.\frac{c^2}{n^2A^2H^2}\,\theta^2$$

$$\text{or}\quad q=\frac{T}{2\pi}.\,\frac{c}{nAH}\,\theta$$

If expressed in coulombs, $q=10.\,\frac{T}{2\pi}.\frac{c}{nAH}\theta$

q can be calculated with the help of this equation if θ is observed and $\frac{cT}{nAH}$ the constant of the instrument is known.

For good working, the period should be large, at least 10 seconds and c small.

MOVING MAGNET INSTRUMENT : A short magnetic needle free to oscillate is placed at the centre of a circular coil as in a tangent galvanometer. If G be the galvanometer constant, *i.e.* the field at the centre of the coil due to unit current in it and i the current through the coil at any instant, the field at the centre of the coil is Gi. If m is the pole strength of the magnet, the force acting on each pole is Gim. If the current flows for a time δt the impulse is $Gmi.\delta t$. The total impulse on each pole when the current lasts for an interval t causing the circulation of charge q, is given by

$$\int_0^t Gmi.dt = Gmq.$$

The moment of the impulse on the magnet of length $2l$ is $Gmq.2l = GMq$, where M is the magnetic moment $(m.2l)$ of the magnetic needle.

If I is the moment of inertia of the magnet about its axis and ω its angular velocity at start, the angular momentum is $I\omega$. By applying second law of motion,

$$I\omega = GMq$$

$$\text{or} \quad \omega^2 = \frac{G^2M^2q^2}{I^2} \quad \ldots \quad \ldots \quad \text{(i)}$$

The work done by the impulse in deflecting the magnet through an angle θ from its position of equilibrium in the magnetic meridian is equal to the initial kinetic energy $\frac{1}{2}I\omega^2$. The work involved is obtained by considering the restoring couple due to earth's field H_0. The couple on the magnet due to H_0 in any deflected position α (with respect to the magnetic meridian) is $MH_0 \sin\alpha$. Total work for deflection θ is obtained as

$$W = \int_0^\theta MH_0 \sin\alpha.d\alpha = MH_0\Big[-\cos\alpha\Big]_0^\theta = MH_0(1-\cos\theta)$$

$$\text{Hence} \quad \tfrac{1}{2}I\omega^2 = MH_0(1-\cos\theta) = 2MH_0 \sin^2\frac{\theta}{2}$$

$$\text{or} \quad \omega^2 = \frac{4MH_0}{I}\sin^2\frac{\theta}{2} \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (i) and (ii) $\frac{G^2M^2q^2}{I^2}=4.\frac{MH_0}{I}\sin^2\frac{\theta}{2}$

$$\text{or} \quad q^2=4.\frac{I}{MH_0}.\frac{H_0^2}{G^2}.\sin^2\frac{\theta}{2} \quad \dots \quad \text{(iii)}$$

For a magnet oscillating in field H_0, the period of oscillation is given by

$$T=2\pi\sqrt{\frac{I}{MH_0}} \quad \dots \quad \text{(iv)}$$

From (iii) and (iv), $q^2 = 4.\frac{T^2}{4\pi^2}.\frac{H_0^2}{G^2}.\sin^2\frac{\theta}{2}$

$$\text{or} \quad q = \frac{T}{\pi}.\frac{H_0}{G}.\sin^2\frac{\theta}{2}$$

This expression determines the charge q which causes a throw θ in terms of H_0 and G.

Damping of oscillations : The motion of a suspended coil or magnet is due to a restoring force proportional to displacement but it also experiences dissipative forces. Hence the motion is never purely simple harmonic, since the amplitude decreases with time. The dissipative forces are due to air resistance, frictional forces, elastic after-effect in the suspension and electro-magnetic damping. As such if θ_1, θ_2, θ_3...be the successive deflections on either side observed at the end of each half period, these are found to be related as shown below,

$$\frac{\theta_1}{\theta_2}=\frac{\theta_2}{\theta_3}=\frac{\theta_3}{\theta_4}= \dots\dots \quad =\delta, \text{ a constant.}$$

δ is called the decrement. Writing $\delta=e^{\lambda}$, we get $\lambda=\log_e\delta$. λ is called the log-decrement. So the true deflection θ_0, as it should have been in the absence of damping factors, is always greater than the observed throw θ. To obtain a relation between θ_0 and θ, we may consider that decrement in a complete oscillation (comprised of two full swings, one each way) may be expressed as

$$\frac{\theta_1}{\theta_3}=\frac{\theta_1}{\theta_2}.\frac{\theta_2}{\theta_3}=\delta^2=e^{2\lambda}$$

So for half-a-swing, $\frac{\theta_0}{\theta}=e^{\frac{\lambda}{2}}=\left(1+\frac{\lambda}{2}+\cdots\right)$ and $\theta_0=\theta\left(1+\frac{\lambda}{2}\right)$

Thus considering the correction for damping the expression for q should be written for the two types of instruments, as

$$q=\frac{T}{2\pi}\cdot\frac{C}{nAH}\cdot\theta\left(1+\frac{\lambda}{2}\right)$$

and $$q=\frac{T}{\pi}\cdot\frac{H_0}{G}\cdot\sin\theta\left(1+\frac{\lambda}{2}\right).$$

where θ is the observed first throw.

Electro-magnetic damping is caused by opposing current induced in the frame of the coil if it is a conducting one. Even when the frame is made of non-conducting material, the motion of the coil itself (when in closed circuit) in the magnetic field induces current in it. This tends to damp the oscillations and bring the coil to rest. The induced *emf* is due to change in flux through the oscillating coil and this fluctuation of flux occurs in the coil even when it rotates in a radial field. This is because of the fact that the field is radial in the air gap, but it is not so inside the iron core (Fig. 2·25). So the flux through the coil changes as it rotates. The flux embraced by the coil (of length l and breadth b) when during oscillation its plane is shifted so as to make an angle θ with its position of rest (parallel to the field inside the core) is the flux passing through the cylindrical surface PQ of length l and width $\frac{1}{2}b\theta$. Hence the change is flux through each turn of the coil is $2H.\ l.\ \frac{1}{2}b\theta=AH\theta$, where A is the area of each turn. If there be n turns, the change in flux $N=AnH\theta$.

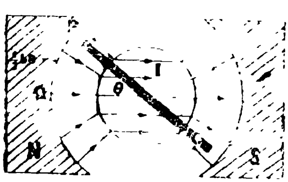


Fig. 2·25

Let R be the total resistance in the galvanometer circuit. The induced current due to the movement of the coil is given by

$$i=\frac{1}{R}\cdot\frac{dN}{dt}=\frac{AnH}{R}\cdot\frac{d\theta}{dt}$$

The potential energy of the equivalent shell measured as the product (strength × flux) is obtained as

$$W = AnH\theta . \frac{AnH}{R} \cdot \frac{d\theta}{dt}$$

The couple acting on the coil due to this current is

$$\frac{dW}{d\theta} = \frac{A^2 n^2 H^2}{R} \cdot \frac{d\theta}{dt}$$

Critical damping resistance : Whether the motion of a suspended system will be periodic or dead beat depends upon the magnitude of the dissipative forces. To make a suspended coil galvanometer suitable for ballistic measurements, the mechanical resistance involved is reduced to minimum and the coil is wound on a non-conducting frame. But yet in a closed circuit the electro-magnetic damping may be large enough to make the motion highly damped one or even aperiodic. This may be remedied by recording the throw in an open circuit. In such a case there is no electro-magnetic damping. If open circuit observation be not possible, the electrical resistance in the galvanometer circuit should be kept so high that the induced current causing the damping is reduced to a low value. Thus electro-magnetic damping can be reduced by proper choice of the electrical resistance (R) in the closed circuit. An oscillatory system is said to be critically damped when the dissipative forces are such that any value greater than this critical value makes the motion non-oscillatory. Even when the system is such that the mechanical resistances are below the critical value, electro-magnetic damping may be considerable. In such a case an electrical resistance in the galvanometer circuit may bring down the dissipative force due to the electro-magnetic effect to the critical value. It is known as the *critical damping resistance.* Any value of electrical resistance less than the critical value (say R) makes the system highly damped or even dead beat. The introduction of such an electrical resistance in a closed galvanometer circuit is necessary for making the instrument function as a ballistic one. It should be noted that an increase in the value of this resistance exceeding this critical value minimises the damping though at the

same time it causes a dimunition of the throw that is of the sensitivity. If R is the critical damping resistance neccssary, it is obtained as the sum of the galvanometer resistance (R_g) and the external resistance (R_x) *i.e.* $R=R_x+R_g$.

It is interesting to observe that for rendering a system periodic *mechanical resistance* should be reduced, but in a coil suspended in a magnetic field for the same purpose over and above the reduction of mechanical resistance the *electrical resistance should be raised to a minimum value* called critical damping resistance.

SENSITIVITY OF A BALLISTIC GALVANOMETER : It is measured as its *quantity sensitivity* (also called figure of merit). It is defined as the deflection observed in millimetres on a scale at one metre distance from the mirror of the galvanometer when there is a discharge of one micro-coulomb of electric charge through the galvanometer.

We have $Q=\frac{T}{2\pi}.\frac{C}{AnH}.\theta\left(1+\frac{\lambda}{2}\right)$

Hence if d is the deflection in millimetres produced by one micro-coulomb *i.e.* 10^{-7} *e.m.* units of charge, then

$$10^{-7}=\frac{T}{2\pi}\cdot\frac{C}{nAH}\cdot\frac{d}{2000}\left(1+\frac{\lambda}{2}\right)$$

$$\text{Quantity Sensitivity } d=\frac{2\pi}{T}\cdot\frac{nAH}{C}\cdot\frac{2000}{1+\frac{\lambda}{2}}\times 10^{-7}$$

If λ is small, $Q.S=4\pi.\frac{nAH}{CT}\times 10^{-4}$ mm/micro-coulomb.

Resistance measurement by the method of damping : The equation of motion of the moving part of a ballistic galvanometer considered as a damped oscillatory system may be expressed as

$$I\frac{d^2\theta}{dt^2}+\left(p+\frac{A^2H^2n^2}{R}\right)\frac{d\theta}{dt}+c\theta=0$$

I is the moment of inertia of the coil, p is the retarding couple per unit angular velocity due to mechanical dissipative

forces, $\frac{A^2H^2n^2}{R}$ is the same due to electro-magnetic effect and c is the torsional couple for unit twist in the suspension strip causing a restoring action.

Putting $\omega^2 = \frac{C}{I}$ and $2k = \frac{P + A^2H^2n^2/}{I}$, the above equation may be written as

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \omega^2\theta = 0$$

The solution of this equation is of the form

$$\theta = Ae^{-kt + \sqrt{k^2 . \omega^2} . t} + Be^{-kt - \sqrt{k^2 - \omega^2} . t}$$

Three cases may arise :

(i) If $k > \omega$, the motion becomes *aperiodic* and the galvanometer is said to be overdamped.

(ii) If $k = \omega$, we have the transitional stage and the galvanometer is considered as critically damped. The deflection is of the form, $\theta = e^{-kt}.(A + B)$

If k is diminished so that ω becomes a bit greater, the motion changes from dead beat to periodic type. Critical damping resistance is obtained by putting $k = \omega$, which means that $p \to 0$, $R = \frac{A^2H^2n^2}{2\sqrt{IC}}$.

(iii) If $k < \omega$, the motion becomes oscillatory though damped. The deflection is obtained in the form,

$$\theta = \theta_0 . e^{-kt} \cos (\sqrt{\omega^2 - k^2})\, t.$$

The period of oscillation $T = \frac{2\pi}{\sqrt{\omega^2 - k^2}}$

It appears from the equation of deflection that the amplitude dies away according to the term e^{-kt},

where $k = \frac{p + A^2n^2H^2/R}{2I}$

Let when $t=0$, the amplitude of oscillation θ_0 be equal to θ_1, so half a period later when $t=\frac{T}{2}=\frac{\pi}{\sqrt{\omega^2-k^2}}$,

$$\text{the amptitude } \theta_2=\theta_1 e^{\frac{-k\pi}{\sqrt{\omega^2-k^2}}}$$

Hence $\frac{\theta_1}{\theta_2}=e^{\lambda}$, where $\lambda=\frac{k\pi}{\sqrt{\omega^2-k^2}}$

Now for a ballistic galvanometer k^2 must be small in comparison with ω^2, hence neglecting k^2.

$$\lambda=\frac{K\pi}{\omega}$$

$$\text{or } \lambda=\left(p+\frac{A^2n^2H^2}{R}\right)\frac{\pi}{2I}\sqrt{\frac{I}{c}}$$

or $\lambda=a\left(\frac{1}{R}+p_1\right)$, where a, p_1 are new constants.

In an open circuit $R\rightarrow\infty$, $\lambda_0=ap_1$ ... (i)

When the galvanometer is short circuited $R=R_g$, the galvanometer resistance, hence

$$\lambda_g=a\left(\frac{1}{R_g}+p_1\right) \quad \ldots \quad \text{(ii)}$$

Again for a total resitance R in the circuit,

$$\lambda_R=a\left(\frac{1}{R}+p_1\right) \quad \ldots \quad \text{(iii)}$$

From (i), (ii) and (iii), $\lambda_g-\lambda_R=a\left(\frac{1}{R_g}-\frac{1}{R}\right)$

$$\text{and } \lambda_R-\lambda_0=\frac{a}{R}$$

$$\text{Hence } \frac{\lambda_g-\lambda_R}{\lambda_R-\lambda_0}=\frac{R-R_g}{R_g}$$

$R-R_g$ is the additional resistance added for the third determination. It may be calculated from the above equation by obtaining λ_0, λ_g and λ_R, the respective log-decrements from actual observations.

But the method is not applicable in practice for when

the galvanometer is short circuited for the second determination it becomes dead beat (since the galvanometer resistance is generally less than the critical damping resistance).

Two high resistances may be compared by this method. If R_1 and R_2 be the resistances included in the second and third experiments in the galvanometer circuit, we obtain the log-decrements as

$$\lambda_0 = ap_1, \text{ in open circuit}$$

$$\lambda_1 = a\left(p_1 + \frac{1}{R_1}\right), \text{ with } R_1 \text{ in circuit}$$

$$\lambda_2 = a\left(p_1 + \frac{1}{R_2}\right), \text{ with } R_2 \text{ in circuit}$$

Hence $\lambda_1 - \lambda_0 = \dfrac{a}{R_1}$, $\lambda_2 - \lambda_0 = \dfrac{a}{R_2}$

So $\dfrac{R_2}{R_1} = \dfrac{\lambda_1 - \lambda_0}{\lambda_2 - \lambda_0}$.

R_g is included both in R_1 and R_2. If R_g *i.e.* the galvanometer resistance be negligible in comparison with R_1 and R_2, the ratio $R_2 : R_1$ may be obtained as shown.

ILLUSTRATIVE EXAMPLE

Find the critical damping resistance for a ballistic galvanometer, having a coil of 300 turns each of area 9 Sq.cm., moment of inertia 4·5 gm-cm² suspended in a field of 1000 oersteds with a strip having torsional constant 2 ergs/radian. Consider the mechanical damping as negligible.

Solution : $R = \dfrac{n^2 A^2 H^2}{2\sqrt{Ic}}$

or $R = \dfrac{300^2 \times 9^2 \times 1000^2}{2 \times \sqrt{4{\cdot}5 \times 2}}$ *e.m. units*

or $R = 1215 \times 10^9$ *e.m.u* $= 1215$ *ohm.*

II-7. FLUX MEASURING INSTRUMENTS

Ballistic galvanometer as flux measurer : If are exploring coil connected in series with a ballistic galvanometer be introduced instantaneously in a magnetic field, there will be an induced *emf* (E) generated in the circuit. If R be the resistance in the circuit, the current flowing is $i=E/R$. If N be the flux passing through the exploring coil then we have,

$$i=\frac{E}{R}=-\frac{1}{R}\cdot\frac{dN}{dt}.$$

If the charge circulated by the flow of current from start to finish is q, then

$$q=\int_0^t i.dt=-\frac{1}{R}\int_0^t \frac{dN}{dt}.dt=-\frac{N}{R}$$

If this charge flowing through the galvanometer (having a constant K) causes a throw θ, then

$$q=K\theta\left(1+\frac{\lambda}{2}\right)=\frac{N}{R}$$

$$\text{or}\quad N=R.K\theta\left(1+\frac{\lambda}{2}\right)$$

So the ballistic galvanometer may be used to determine flux. If α be the area of the exploring coil and H the field in which it is introduced, $N=\alpha H$.

Thus the exploring coil with help of a ballistic galvanometer may determine the strength of a magnetic field. But the method has the restriction that the change of flux causing the throw must cease before the deflecting system has moved. This is a disadvantage since an exploring coil cannot be so quickly inserted inside the magnetic field under investigation. This, however, may be avoided in special experimental methods.

Grassot Fluxmeter : A special type of moving coil instrument designed to measure magnetic flux is known as fluxmeter. Fluxmeter has this advantage over the ballistic galvanometer that the change of flux need not be instantaneous.

Grassot Fluxmeter (Fig 2·26) consists of a fairly large

rectangular coil (C) suspended from a single silk fibre which exerts very little torque. There is thus no restoring force and the coil remains at rest in any deflected position. Air damping is negligible. The electro-magnetic damping which is designed to be fairly large has an important function. The current is carried to the coil by two light silver springs (s) offering very little damping. Field is obtained from a

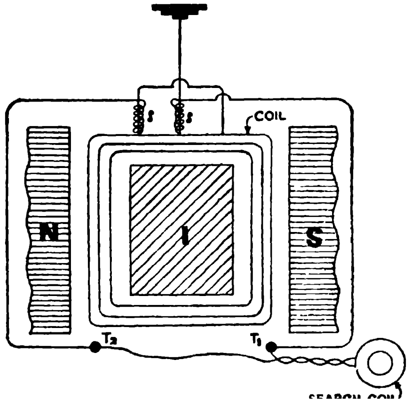


Fig. 2·26

permanent magnet with (NS) concave cylindrical surfaces. A co-axial soft iron cylinder (I) is placed inside the coil to make the field radial.

The terminals attached to the springs are connected to the exploring coil. When this is introduced in a magnetic field, the change of flux causes a deflection in the suspended system. If the effective area of the coil be a, the flux through this when in a field of induction B is $N=Ba$. In air medium $B=H'$, so $N=aH'$, H' is the field to be explored.

During introduction in the field the flux through the exploring coil changes and there is an induced *emf* $\frac{dN}{dt}$. When the coil of the fluxmeter moves in the field inside there is also an induced *emf* in it. If A be the effective area of all the turns of the coil, H the field due to the permanent magnet and $\omega=\frac{d\theta}{dt}$ the angular velocity of the coil, the induced *emf* is $AH.\frac{d\theta}{dt}$. Further, there may be a small *emf* as the current in the coil grows due to self-inductance of the exploring coil, fluxmeter coil and the current leads. This may be represented as $L\frac{di}{dt}$. Hence the effective *emf* in the fluxmeter coil as the

exploring coil is being introduced in the field under investigation is given by

$$E = \frac{dN}{dt} - AH.\frac{d\theta}{dt} - L\frac{di}{dt}.$$

If R is the resistance of the circuit including those of the exploring coil, the fluxmeter coil and the external resistor, then the current i is obtained as

$$i = \frac{E}{R} = \frac{1}{R}\left[\frac{dN}{dt} - AH.\frac{d\theta}{dt} - L\frac{di}{dt}\right]$$

The fluxmeter coil carrying a current i being in a magnetic field H experiences a couple iAH. Let I be the moment of inertia of the coil. While moving with an angular acceleration $\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$ it has the moment of momentum $I.\frac{d\omega}{dt}$. Further, let

$\alpha.\frac{d\theta}{dt}$ be the couple due to mechanical damping. Equating the moment of momentum with the effective couple producing the motion, we may write

$$I\frac{d\omega}{dt} = iAH - \alpha\frac{d\theta}{at}$$

Substituting for i, we have

$$I\frac{d\omega}{dt} = \frac{AH}{R}\left[\frac{dN}{dt} - AH\frac{d\theta}{dt} - L\frac{di}{dt}\right] - \alpha.\frac{d\theta}{dt}$$

Integrating from start to finish with respect to time t

$$\int_0^t I\frac{d\omega}{dt} = \frac{AH}{R}\left[\int_0^t \frac{dN}{dt}dt - AH\int_0^t \frac{d\theta}{dt}dt - L\int_0^t \frac{di}{dt}dt\right] - \alpha\int_0^t \frac{d\theta}{dt}dt$$

$$\text{o} \quad [I\omega]_0^t = \frac{AH}{R}[N]_0^t - \frac{A^2H^2}{R}[\theta]_0^t - L\frac{AH}{R}[i]_0^t - \alpha[\theta]_0^t$$

Since there is no restoring force, $\omega = 0$, both at the start and finish and since the current i which is totally induced ceases when the motion of the coil is stopped, it is also zero at the start as well as in the end. We may write the above equation if θ be the angular shift of the coil, as

$$0 = \frac{AH}{R}\left[N - AH\theta\right] - \alpha\theta$$

If α becomes negligible, $N=AH\theta=K\theta$, K being a constant. N is the total change in the flux through the search coil. This relation is independent of the time during which the flux changes. Thus to measure a flux the search coil is to be introduced inside the field and the change in position of the fluxmeter coil is to be observed. A pointer moving over a calibrated scale or the deflection of a spot of light reflected from a a mirror attached to suspension may be used to determine θ. Since $N=K\theta$, K should be obtained from the observed deflection for a known flux. The field explored is $H'=N\alpha=K\alpha\theta$.

Note : In a dead beat galvanometer *mechanical control* determines the steady deflection and the *electro-magnetic damping* is utilised to make it aperiodic.

In a fluxmeter there is no restoring action or *mechanical control, electro-magnetic damping* exclusively produces the control which however ceases when the deflecting couple vanishes.

In a ballistic galvanometer *mechanical control* determines the throw and the *electro-magnetic damping* which tends to reduce the throw is designed to be minimum.

II-8. ELECTRO-DYNAMOMETERS

Instruments based on mutual action of currents : The moving system of dynamometer type of instruments are conductors free to move in a field due to another fixed system of conducters, both of these being traversed by current. The conductors usually in the form of coils may be arranged in different ways but for absolute measurement the design of the system should be such that the constants of the instruments may be directly obtained.

Kelvin's Ampere Balance : This is a current measuring device based on mutual action between conductors carrying current. Four coils (Fig. 2·27) A, B, C, D are electrically connected in series and they carry the same current. These coils are fixed and between these coils (arranged in two pairs) there are two movable coils (E, F carrying the same current as the fixed coils. These are attached to the opposite ends of a balance beam which is hung at its centre from two braids

each made up of a number of very fine copper wires which also serve as current leads. The currents in the separate coils

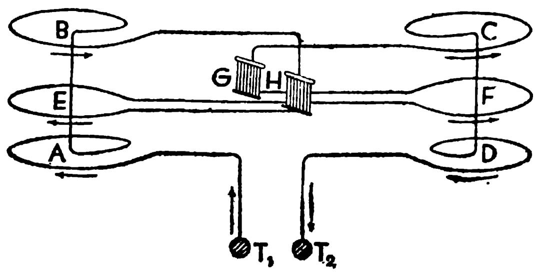


Fig. 2·27

are so directed that if the force due to *A-B* on *E* urges it downwards, the force on *F* due to *C-D* is directed upwards.

The couple due to the coils causes a deflection of the beam which may be restored to its horizontal position again by shifting a sliding weight along the beam. There are counterpoises to keep the beam horizontal when no current flows and the corresponding sliding weight is then at zero mark at one end. The couple due to current in coil varies as square of current (i^2) and the couple due to the weight is proportional to displacement (d) and hence

$$i^2 \propto d$$

$$\text{or} \quad \pm i = K\sqrt{d}$$

The displacement of the sliding weight is independent of the direction of the current and so the instrument is suitable for measurement of alternating current.

If W be the mutual potential energy of the two coils (say A-E) and i is the current in each, then $W = Mi^2$, where M is the mutual inductance of the two coils.

Force between the two coils is $F = \dfrac{\partial W}{\partial x} = i^2 \dfrac{\partial M}{\partial x}$

If a is the length of the arm of the beam and d is the shift of a mass m, then considering that there are four coils

$$4F.a = mg.d$$

$$\text{or} \quad F = \frac{mg.d}{4a} = i^2 . \frac{\partial M}{\partial x}$$

$$\text{or}\quad i^2 = \frac{mgd}{4a} \bigg/ \frac{\partial M}{\partial x}$$

$$\text{so}\quad i \propto \sqrt{d},\ \text{or}\ i = K\sqrt{d}$$

In order to obtain the value of current directly from the shift of the weight, the beam is calibrated. For this the value of K is to be known. This is obtained by putting a silver voltameter in series with the coils. The value of i being known, K is calculated from the observed value of d. The instrument is calibrated and a fixed scale is attached to the beam which is marked in amperes. To alter the range there are several sliding weights and for each such weight there is a corresponding counterpoise to be placed at the other end of the beam.

Siemen's Electro-dynamometer : It consists of two coils, one of which is fixed and the other movable. The movable coil is suspended by a silk fibre. When there is no current the planes of the two coils remain at right angles to each other. One end of a spring (S) is attached to the movable coil and the other end to a torsion head (T) which carries a pointer (P) moving over a horizontal circular scale. The ends of the movable coil (AB) dip in two mercury cups placed one above the other. Another pointer p is attached to the movable coil, its range of deflection being limited by stops. The instrument is so placed that the plane of the moving coil is perpendicular to the magnetic meridian. This is for elcminating the effect of earth's field. The pointer p should in this position indicate zero. The current to be measured is made to circulate in the two coils in series. The mutual forces tend to bring the movable coil in the same plane as the fixed coil. The torsion head with the pointer P is now turned in opposite direction until p is again at zero.

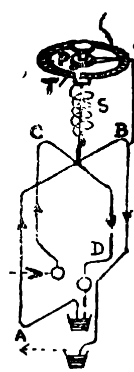


Fig. 2·28

If i be the current and θ the angle through which P has been rotated, then

Couple between the coils $\propto i^2$

Couple due to torsion$=c\theta$, where θ is the torsional couple for unit twist.

Hence $i^2 \propto c\theta$, or $i=K\sqrt{\theta}$, K is a constant to be determined by passing a known current. Since the rotation is proportional to square of current this instrument may be used for measurement of alternating current also.

Weber's Electro-dynamometer : A fixed pair of Helmholtz coils (AB) produces a uniform magnetic field between them on their common axis and a small coil (C) free to rotate is suspended at the mid-point between the fixed coils. When there is no current through the coils (which are all in series) the plane of the movable coil is at right angles to the planes of the fixed coils, which are placed in the magnetic meridian. When a current is passed the movable coil rotates tending to bring its plane parallel to each of the fixed coils. The deflecting couple is balanced by the restoring couple due to earth's field (H_0) and due to twist in suspension strip. The magnetic moment (M) of the suspended coil of effective area A is Ai, when a current i flows through it. If G is the field due to unit current in the fixed coils at a point midway between them on the common axis and if θ_1 be the deflection of the movable coil then by condition of equilibrium of a magnet in mutually perpendicular fields,

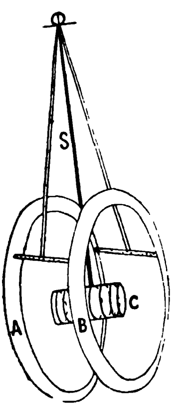


Fig. 2·29

$$MGi\cos\theta_1 = MH_0\sin\theta_1 + K\sin\theta_1$$

($K\sin\theta_1$ is the restoring couple due to twist in suspension)

$$\text{or} \quad \tan\theta_1 = \frac{MGi}{MH_0+K} = \frac{MGi/K}{1+\frac{MH_0}{K}}.$$

$$\text{or} \quad \tan\theta_1 = \frac{MGi}{K}\left[\left(1+\frac{MH_0}{K}\right)^{-1}\right] = \frac{MGi}{K} - \frac{M^2GH_0i}{K^2} \ldots \ldots \text{(i)}$$

Again if θ_2 be the deflection on the reversal of the current,

then, $$\tan\theta_2 = \frac{MGi}{K} + \frac{M^2GH_0i}{K^2} \qquad \text{......(ii)}$$

Adding (i) and (ii),

$$\tan\theta_1 + \tan\theta_2 = \frac{2MGi}{K} = \frac{2AGi^2}{K}$$

$$\text{or} \quad i^2 = \frac{K}{2AG}\,(\tan\theta_1 + \tan\theta_2)$$

$$\text{or} \quad i = \beta\sqrt{\tan\theta_1 + \tan\theta_2}$$

β is to be obtained by determining the deflections with known current. The instrument is suitable for measurement of alternating current. In modern form of apparatus the moving coil instead of being suspended is pivoted and the deflection is observed by means of a pointer moving over a square-law scale.

Kelvin's Watt Balance : The instrument contains four fixed coils connected in a series circuit, which includes the appliance (say a glow lamp), the power in which is to be obtained. There are two movable coils (*E-F*) at the ends of a balance beam suspended at its mid-point. The ends of these

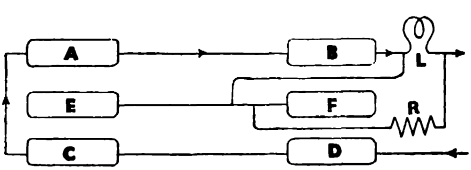


Fig. 2·30

movable coils (Fig. 2·30) are connected in series and these are joined through *R*, a high resistance coil to the terminals of the appliance (*L*) concerned. The currents in the coils are so arranged that by mutual action one pair of coils urges one movable coil upwards and the other drives the other coil downwards. The beam is deflected in one direction and is brought back in position again by weight sliding on the beam.

Let the total resistance of the movable circuit be R and E the potential drop at the ends of the lamp terminals, then the current in the movable coil is E/R. Hence the force on each movable coil is Ei/R, where i is the current through the lamp as well as in the fixed coils. If the force acting on the movable coil is balanced by a displacement d of the weight, then

$$\frac{Ei}{R} \propto d, \quad \text{or} \quad Ei = Kd$$

Since Ei is the power absorbed in the lamp, it may be obtained by calibrating the beam in a scale in watts by separately using ammeter and voltmeter in the two circuits.

II-9. SPECIAL TYPE INSTRUMENTS

Einthoven String Galvanometer : A thin wire of tungsten (phosphor-bronze, silver, copper, quartz or silvered glass fibre may also be used) is stretched taut across a strong magnetic field in a narrow gap. When the wire carries a current, it is acted upon by a force, at right angles, both to the direction of the current and the magnetic field, according to Fleming's left hand rule. The deflection of the wire is observed through a microscope inserted in a hole in the magnet, the wire being illuminated by a beam of light coming through another hole in the other polepiece. The eyepiece of the microscope contains a graticule divided in linear scale and the deflection of the wire is measured in it. Current of the order of 10^{-5} micro-ampere can be measured with this instrument and it responds to transient and alternating currents of frequencies upto 200 cycles per second. The A-C wave form may be recorded in a film by projecting the shadow of the fibre by a camera objective to from a point on the film behind it. If the film is made to run in a direction

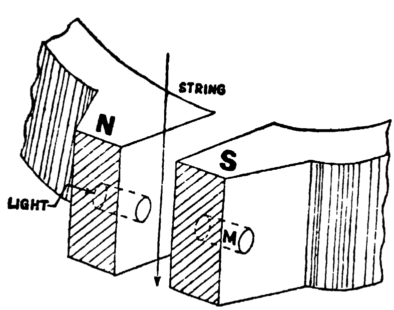


Fig. 2·31

perpendlcular to the direction of movement of the string, the oscillatory wave form may be recorded photographically.

Campbell Vibration Galvanometer : A d'Arsonval type of galvanometer with a moving coil (*C* in Fig. 2·32) having a few turns is suspended between the pole pieces of a strong permanent magnet. The suspension is bifiliar above and below the coil serving also as current leads to the coil. The length of the suspension may be altered by a bridge piece (*B* in Fig. 2·32) under the wires and its tension may be changed by a spring (*Sp*). Thus the frequency of vibration of the coil may be tuned to different frequencies of supply current. Further since the moving system has small damping, resonance is sharp amounting to high selectivity.

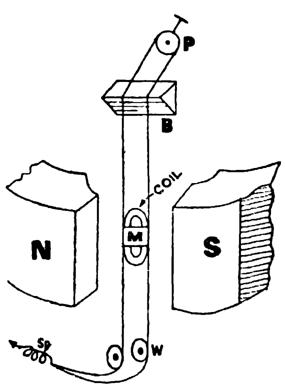


Fig. 2 32

When an alternating current is passed through the coil the magnetic field causes it to vibrate, the amplitude being greatest at the particular frequency (of the current) to which the coil has been adjusted for. The deflection is greatly magnified at resonance, hence the arrangement becomes extremely sensitive even to weak alternating currents. The moving system carries a small mirror (*M*) and when the coil vibrates an extended band of light appears on the scale. This type of galvanometer is a suitable apparatus for detection of null condition in A-C bridge experiments.

II-10. DIRECT READING INSTRUMENTS

Ammeter : The deflection in a galvanometer depends on the current flowing through it. A scale calibrated in amperes may be attached to a galvanometer provided with a pointer fitted with the coil. Such an instrument is called an ammeter.

It is generally provided with a low resistance coil which is pivoted between two needle points and its movement is

is controlled by hair-springs which also serve to conduct the current to and from the coil. The coil is in a strong radial magnetic field. It has an attached pointer which moves over a scale graduated in amperes.

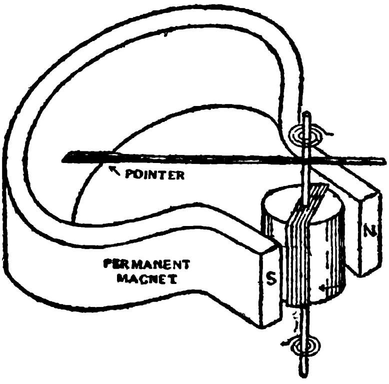


Fig. 2·33

The coil is generally shunted by a small resistance so as to keep the resistance of the instrument low. It ensures that the inclusion of the ammeter does not disturb the current to be measured. The shunt also allows variation of the range of the instrument. If i be the current to be measured and i_g be that flowing through the ammeter coil, then

$$i_g = i.\frac{S}{S+G},$$

S and G being respectively the shunt and galvanometer resistances.

The current to be measured is $\frac{S+G}{S}$ times the current indicated by the instrument. $\frac{S+G}{S}$ is called the *multiplying power* of the shunt. If S be small in comparison with G, this power may be represented by the quantity G/S.

If the current to be measured is increased n times, the current discussed in the above *i.e.*, if it is now ni, the shunt S may be changed to such a value as to send the same current as before through the coil. For this the value of the shunt will be $\frac{1}{n}$th of its former value. So a change of shunt multiplies the working range of the instrument. Thus the same instrument (comprising particular coil and magnet) may be used for different ranges if it is provided with a set of

shunts. As shown in sketch (Fig. 2·34) the connection is made to the positive terminal and to one of the three terminals indicated as High, Medium and Low meant for corresponding current strengths. Although the coil receives only a fraction of the total current, the scale is graduated to show the full value of the current. Thus an instrument with variable shunts has separate negative terminals and separate scales, each one for a limited range.

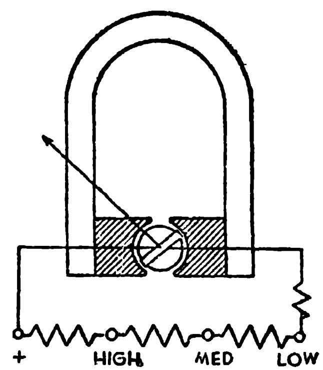


Fig. 2·34

Voltmeter : The components of a voltmeter are same as that of an ammeter. But instead of a low resistance shunt there is a high resistance coil in series with the galvanometer coil. A voltmeter while in use is placed in parallel with the circuit element. To avoid any alteration of the circuit condition due to introduction of the voltmeter, very small current should be diverted through the instrument, so the latter should possess a high resistance. The resistance in series with the voltmeter is called *multiplier*. The multiplier if variable increases the range of the instrument. If the series resistance is increased n times an increased (n times) potential difference at the terminals produces the same current through the coil and so the same deflection. In this way a voltmeter may be modified to multiply the working range. Actually a multiplier permits the connection of a multiplied voltage at the ends of the coil of the instrument.

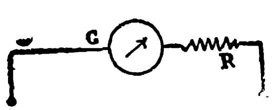
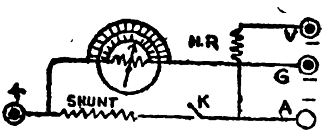


Fig. 2·35

Avometer : The same instrument may be used either as a voltmeter or as an ammeter. If the coil is used with a shunt it serves as an ammeter. By taking off the shunt resistance and connecting a high resistance in series with the coil the ammeter is converted into a voltmeter. As shown in the

sketch (Fig. 2·36) when the terminals + and G are used the instrument is an ordinary galvanometer. It serves as a voltmeter when the terminals + and V are used. To use it as an ammeter, the connections are made with + and A, the key K being closed. There are separate scales for different uses.



Fig, 2·36

Ohm-meter : In a common type of instrument resistance measurement involves the production of current by application of a fixed potential difference at the galvanometer coil terminals having the resistance to be measured in series with coil. The greater the resistance the smaller is the deflection. The graduations of the scale to read ohms therefore proceed in the reverse way relative to the scale for current measurement.

Soft Iron Instruments : In these type of instruments the current to be measured is passed through a fixed coil. The magnetic field produced by the current flowing through the coil is arranged to produce movement in a piece of soft iron. There are two types of instruments.

In **attraction type instrument** a small piece of iron in the form of a small disc is pivoted eccentrically inside the coil (Fig. 2·37a). When the coil carries a current the iron is magnetised and is attracted towards the interior of the coil. The motion which is proportional to the magnetic flux inside the coil is controlled by a restoring torque due to gravity or a hair spring.

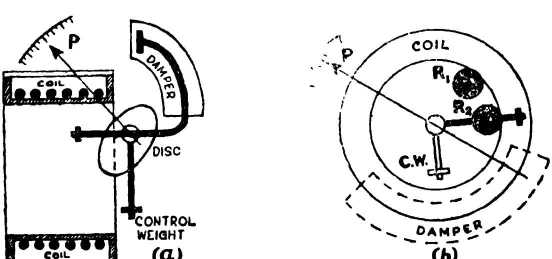


Fig. 2·37

In **repulsion type instrument** the two rods of soft iron placed inside the coil (Fig. 2·37 b). One is fixed (R_1) and the other

(R_2) is free to turn about a pivot. When a current flows through the coil the two rods are similarly magnetised and as such repel one another, the pivoted rod turning against the restoring action due to gravity or hair spring.

In both the instruments damping is obtained from a dash pot arrangement in which a light plunger moves in a tube, the movement being opposed by viscosity of air. This does not, however, make the instrument dead beat but stops the oscillation in a short time.

The force action is proportional to the square of the current, since magnetisation and magnetic field are both proportional to current. Thus *the moving iron instruments are suitable for alternating current* circuits and are widely used for this purpose. The value of the current is obtained from the calibrated dial over which moves a pointer attached to the moving system.

Shunts and multipliers are inserted for the use of the instrument as ammeter and voltmeter respectively. Resistances used for this purpose should be non-inductive.

Main disadvantages of this type of instruments are the action of external magnetic fields and effect of hysteresis. These are remedied by adequate shielding and using magnetic alloys (instead of soft iron) with low hysteresis.

ILLUSTRATIVE EXAMPLES

1. *A galvanometer coil of resistance 80 ohms being shunted with a resistor of 10 ohms is connected to a battery of emf 2 volts through an external resistance 200 ohms. Calculate the galvanometer current.*

Solution : $$i=\frac{E}{R+\frac{SG}{S+G}}=\frac{2}{200+\frac{80\times 10}{80+10}}=\frac{2\times 9}{1880}$$

$$i_g=i.\frac{S}{S+G}=\frac{2\times 9}{1880}\times\frac{10}{90}=\frac{2}{940}\text{ amp.}$$

2. *The moving coil of a galvanometer has 60 turns each of width 2 cms. and length 3 cms. Find the torque when it carries a current of 1 milli-ampere if the field inside be 500 oersteds.*

Solution : Couple$=nAiH=60\times2\times3\times1\times10^{-4}\times500$

or $C=18$ dyne-cm.

3. *A galvanometer whose needle deflects 5 scale divisions per mA. current is to be used as an ammeter. Its coil resistance is 500 ohms. What should be the resistance of the shunt in order that the needle may deflect 10 divisions per ampere ?*

Solution : 10 div. per amp.$=5$ div. per 0·5 amp.

Using the formula $i_g=i\dfrac{S}{S+G}$

$$\frac{i_g}{i}=\frac{S}{S+G} \text{ or } \frac{i}{i_g}=\frac{S+G}{S}=1+\frac{G}{S}$$

$$\text{or } \frac{G}{S}=\frac{i}{i_g}-1=\frac{i-i_g}{i_g} \text{ or } S=\frac{G.i_g}{i-i_g}$$

We have $G=500$ ohms, $i_g=10^{-3}$ amp. $i=0{\cdot}5$ amp.

$$\text{So} \quad S=\frac{500\times10^{-3}}{0{\cdot}5-0{\cdot}001}=\frac{0{\cdot}5}{0{\cdot}499}=1 \text{ ohm (nearly)}$$

4. *A milli-ammeter of resistance 20 ohms reads upto 0·05 amp. Obtain how the instrument may be converted into (a) an ammeter to read upto 5 amps. and (b) a voltmeter to record upto 50 volts.*

Solution : (a) Let the value of the shunt be S. The current through the coil of resistance 20 ohms is 0·05 amp. and the current through the shunt is 5—0·05=4·95 amp. Equating the potential drops at the ends of the two resistances,

$$V_a-V_b=i_g.G=i_s.S$$

So $0{\cdot}05\times20=4{\cdot}95.S$, hence $S=0{\cdot}202$ ohm.

(b) Let R be the high resistance in series

$$i=\frac{V}{R+G}$$

$$\text{or } 0{\cdot}05=\frac{50}{R+20}, \text{ hence } R=910 \text{ ohms.}$$

EXERCISES ON CHAPTER II

2-1. State and explain Laplace's law for the magnetic field due to a current element at a distant point. Hence obtain the magnetic field near an infinitely long straight conductor carrying current.

2-2. Enunciate Ampere's theorem and obtain it in the circuital form.

Obtain the field at any point on the axis of a circular coil.

2-3. What is an equivalent shell? A circular coil of radius r carrying a current of 10 amperes is placed in the magnetic meridian with its plane horizontal. Obtain the couple acting on the coil.

[*Ans.* $\pi r^2 . H_0$]

2-4. Find an expression for the field at any point inside a solenoid of finite length. Hence or otherwise obtain the field inside an anchor ring.

2-5. Prove that the force of an element δs carrying a current i when placed in a magnetic field of induction B is given by $i\,[\delta\mathbf{s} \times \mathbf{B}]$

2-6. Find the force due to a small magnet of moment i, the plane of the coil being normal to the axis of the magnet and its centre on the axis at a distance r from the mid-point of the magnet.

$$\left[\textit{Ans.} \quad \frac{6\pi Mi\ a^2\sqrt{r^2-a^2}}{r} \right]$$

2-7. Two similar coils are placed coaxially at a distance apart equal to their radius. What is the nature of the magneic field at a point on the axis mid-way between the coils? Describe how this property has been utilised in Helmholtz form of tangent galvanometer.

2-8. Obtain the mechanical force between two parallel co-axial coils carrying current when one of the coils is very small compared with the other.

2-9. Two circular coils of radii R and $(R+a)$ lie coaxially in parallel planes, separated by distance d. Discuss the force between them when same current flows through both the coils.

2-10. Describe the working of a suspended coil galvanometer. How is it made dead beat? Define its figure of merit. Discuss the relation between current sensitivity and voltage sensitivity. What are the factors determining current sensitivity?

2-11. Explain the conditions under which a moving coil type of galvanometer becomes ballistic. Give the theory of

the ballistic galvanometer. How would you apply corrections for damping ?

2-12. What is electro-magnetic damping ? Explain the term critical damping resistance. What are the factors determining it ?

Discuss how the damping of such a galvanameter may be utilised for comparison of high resistances.

2-13. Work out the theory of the Grassot Fluxmeter and compare its working with that of Ballistic Galvanometer.

2-14. How many turns should the coil of a ballistic galvanometer be provided with so that its critical damping resistance may be 1120 ohms, the area of the coil is 6 sq. cms and its moment of inertia is 1·5 gm.-cm^2, the field strength is 1000 oersteds and the torsion constant is 2 ergs/ radian.

[*Ans* : 300]

2-15. Describe the construction and working of Kelvin's Ampere balance.

2-16. A small coil of 125 turns and mean radius 2 cm. is suspended with its centre midway between a pair of Helmholtz coils, each of radius 20 cm. and 200 turns of wire. If a current of 0·5 amp. passes through all the coils, find the couple on the small coil when the plane is normal to that of either of the larger coils.

[*Ans* : 353 dyne-cm]

2-17. Write notes on : (a) Einthoven String Galvanometer. (b) Campbell Vibration Galvanometer.

2-18. Explain the working of an ammeter. Discuss how an ammeter may be converted into a voltmeter.

2-19. You are given a milli-voltmeter of range 1-30 millivolts and of internal resistance 25 ohms. Show how you would use the instrument to measure (a) potential difference between 1 30 volts and (b) currents between 0·1—3 amp.

[*Ans* : (a) Series resistance 24975 ohm
(b) Shunt resistance 0·01 ohm.]

2-20. What are soft iron instruments ? Describe in brief their working principles. What are their special uses ?

CHAPTER III

ELECTRO-THERMAL AND THERMO-ELECTRIC EFFECTS

III-1. JOULE'S HEAT PRODUCTION

Relation between Current and Heat generation: Joule experimentally obtained that heat generated in a given time in a particular conductor is directly proportional to the square of the current. Joule's experimental law may be deduced from theoretical considerations.

Let a charge δq coulomb pass between two points A-B in a resistor maintained at a difference of potential $(V_a - V_b)$ volts. The work involved is $\delta W = (V_a - V_b).\delta q$ joules. If this charge is carried by a current i amperes in time δt, $\delta q = i.\delta t$. If e represents $(V_a - V_b)$ then $\delta W = e.\delta q = ei.\delta t$. The energy developed in the resistor in time t is $W = eit$ joules. If this energy is not converted into any other form, it generates heat inside the resistor and the amount of heat H in calories is obtained from the equation $W = JH$, hence

$$W = eit = JH$$

$$\text{or} \quad H = \frac{eit}{J} \text{ calories}$$

J, being the mechanical equivalent of heat. If r is the resistance of the conductor between the points, then $e = ir$ and so

$$H = \frac{i^2rt}{J} \text{ calories}$$

If i and r are measured in electro-magnetic units $J = 4{\cdot}2 \times 10^7$ ergs/calorie and if these are calculated in practical units $J = 4{\cdot}2$ joules/calorie.

Principle of least heat: Current in several branches in a circuit is always so distributed that the production of heat is minimum. Let two resistances r_1 and r_2 be joined in parallel,

out of the total current i, a portion i_1 flows through r_1 and current through r_2 is $(i_2 - i_1)$. The total heat production per unit time expressed in joules in two resistors is obtained as,

$$H = i_1^2 r_1 + (i - i_1)^2 r_2$$

$$\frac{dH}{di_1} = 2i_1 r_1 - 2(i - i_1) r_2$$

$$\frac{d^2H}{di_1^2} = 2r_1 + 2r_2$$

$\frac{dH}{di_1}$ becomes zero if $i_1(r_1 + r_2) = ir_2$ and $\frac{d^2H}{di_1^2}$ is $+ve$.

The rate of heat production is minimum when this condition is satisfied, *i.e.* when

$$i_1 = \frac{i}{r_1} \cdot \frac{r_1 r_2}{r_1 + r_2} = \frac{Ri}{r_1},$$ R is the equivalent resistance.

Similarly it may be obtained that $i_2 = R\frac{i}{r_2}$

Thus H becomes minimum when current in each circuit is inversely proportional to its resistance. This however is the same as the law of distribution of current obtained by application of Ohm's law. Hence we get the principle that *in branching of current in several branches, the distribution is such that the heat production is minimum.* The principle may be shown to be true in a generalised form, that is in a network.

Maximum power theorem : Let a battery of *emf* E and internal resistance r be connected through an external resistance R to form a closed circuit. The current through R is obtained as

$$i = \frac{E}{R + r}$$

The energy dissipated per second in R is given by

$$W = i^2 R = \frac{E^2 R}{(R + r)^2}$$

Hence $$\frac{dW}{dR} = \frac{E^2(R + r)^2 - E^2 R.2(R + r)}{(R + r)^4}$$

If W be either maximum or minimum, $\frac{dW}{dR} = 0$

$$i.e. \quad (R+r)^2-2R(R+r) = 0$$

$$\text{or} \quad (R+r) = 2R$$

$$\text{or} \quad R = r.$$

Since $\frac{d^2W}{dR^2}$ becomes negative for this value of R, W is maximum.

Thus *energy dissipated in an external circuit due to current drawn from a battery becomes maximum when the external resistance is equal to the internal resistance.*

Electrical method of determination of Joule's equivalent : An accurate method of determination of mechanical equivalent of heat (J) is due to *Callendar and Barnes* in a *Constant flow calorimeter.* Water flows steadily along a glass tube and it is heated electrically by a wire carrying a constant current. Outside the glass tube there is a jacket and the space between the calorimetric tube and the jacket is kept vacuum. A water jacket surrounds the vacuum. Two thermometers (T_1, T_2) measure the temperatures (θ_1, θ_2) of incoming and outgoing water. The current (i) flowing is measured in amperes by a potentiometer using a standard cell and a standard resistance. The potential drop (e) at the ends of the wire is measured in volts also by potentiometer. The mass (m) of water in grammes flowing per second is obtained. The thermometer reading in the steady state gives the rise of temperture $\theta=(\theta_2-\theta_1)$ in degrees celsius.

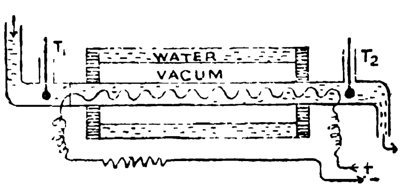


Fig. 3·1

A calorimeter of this kind does not absorb heat when the temperatures are steady. In such a case the electrical energy expended per second ($e.i$) is used in warming the flowing water and in making up for the radiation loss, which is, of course, low.

If h is the loss of heat by radiation and s is the specific heat of water, then heat taken by water is $ms\theta$, so

$$\frac{ei}{J}=ms\theta+h \quad \ldots \quad (i)$$

To eliminate h, a second set of observations is taken by slightly altering the current and adjusting the flow of water to keep θ_2 and θ_1 unchanged. If e', i' and m' are the corresponding data for the second experiment, then

$$\frac{e'i'}{J}=m's\theta+h \quad \ldots \quad (ii)$$

From (i) and (ii), $J=\dfrac{ei-e'i'}{(m-m')s\theta}$ joules/calorie.

For correct elimination of h by this method thermometers should be sensitive enough to record temperature to $\frac{1}{20}$th of a degree.

III-2. THERMO-ELECTRIC EFFECTS

Seebeck effect : In 1821 Seebeck discovered a phenomenon in which thermal energy is found to be directly transformed into electrical energy. He observed that when a circuit made of two dissimilar metals has the two junctions kept at unequal temperatures a current flows through the circuit. A simple circuit is shown in diagram (Fig. 3·2 a), in which an iron strip is joined at its ends to pieces of copper through a low resistance galvanometer in series. By keeping one of the junctions at 0°C and raising the temperature of the other, a current is found to flow from copper to iron across the hot junction and from iron to copper through the cold junction. The current rises with increase of temperature of the hot junction till it is at 275°C, when the current becomes

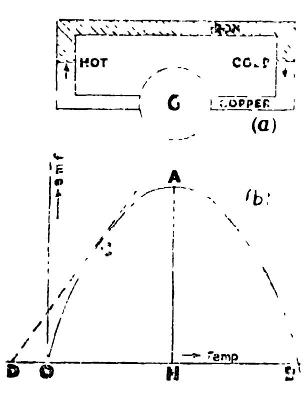


Fig. 3·2

maximum. For every pair of metals there is such a temperature of the hot junction for which the current is maximum. This temperature is known as **Neutral temperature.** If the hot junction temperature is further raised the current decreases and for the iron-copper couple it becomes zero when the hot junction is at about 550°C. At temperatures above this the current increases again but this time in the reversed direction. The temperature at which the current becomes zero before reversal is called the **Inversion temperature** of the particular couple. This temperature is always as much above the neutral temperature as the cold junction is below it. It is not a constant but depends upon the cold junction temperature. The *thermo-electric curve i.e.* the graph of *emf* against the temperature difference of the junctions (Fig. 3·2b) is generally obtained as a parabola, but this relation between *emf* and temperature is approximately true.

Current produced in this way by the heating of one of the junctions is known as *thermo-electric current*, the pair of metals forming a thermo-electric circuit is called a *thermo-couple*, the phenomenon being known as *Seebeck effect.* Seebeck arranged the metals in a series ; the current flows across the hot junction from the metal occuring earlier in series to one coming latter. The separation of the metals in the series determines the magnitude of the *emf* to be expected when any two of these are used as a thermo-couple.

SEEBECK SERIES : Bi,Ni,Co,Pd,Pt,Ur,Cu,Mn,Ti,Hg,Pb,Sn,Cr. Mo,Rh,Ir,Au,Ag,Zn,W,Cd,Fe,As,Sb,Te.

According to the rule stated the Bismuth-Antimony pair form a thermo-couple giving almost the maximum thermo-emf available. In this couple the current flows from antimony to bismuth across the cold junction.

THERMO-ELECTRIC POWER : If δE is the small *emf* generated in a thermo-couple due to a small temperature difference δT between the junctions (originally both being at the

same temperature T), then Lt.$\frac{\delta E}{\delta T}$ *i.e.* $\frac{dE}{aT}$ is known as the thermo-electric power of the thermo-couple at the temparature T. The quantity $\frac{dE}{dT}$ which may be obtained for any temperature (T) of the hot junction (in absolute scale) from the slope of the thermo-electric curve (Fig. 3·2b) is the thermo-electric power at that temperature. A graph of $\frac{dE}{dT}$ plotted against T is the *thermo-electric line.*

Peltier effect : If in a circuit made with a thermo-couple instead of supply of heat at one junction (to keep it hot) and absorption at the other (to keep it cold) current is sent through the junctions from an external source of *emf* included in the circuit, it is found that one of the junctions gets heated and the other loses heat. If the source of *emf* sends current through the junctions in the same direction as that would be due to Seebeck effect, the junction that would receive heat for generation of thermo-current now absorbs heat and is thereby cooled and the junction that would be kept at a lower temperature now gets heated, due to the passage of current through the junctions in the same direction (*cf.* Figs. 3·3a and 3·3b). This discovery was made by Peltier and is known as *Peltier effect.*

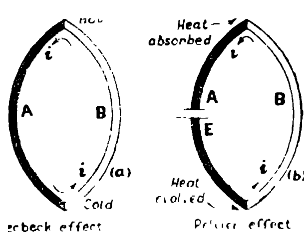


Fig. 3·3

The heat generation and absorption at the two junctions are interchanged if the direction of current is reversed. It should be noted that Peltier heat production is different from Joules' heat production, which as distinct from Peltier effect is a non-reversible effect.

The Peltier heating effects may be demonstrated by an arrangement described in the next page.

Two identical fine copper coils are wound round the two junctions of an antimony-bismuth couple. The coils form respectively the two arms of a Wheatstone bridge arrangement. The bridge is set to be in a balanced condition when there is no current in the thermo-electric circuit. If a current is now passed through the junctions, one of these gets heated and the other is cooled. This causes change in the resistance of the coils and the balance of the bridge is disturbed. To restore the balance of the bridge it would be found that the resistance of one coil decreases and that of the other increases and these effects are reversed by reversal of current through the junctions.

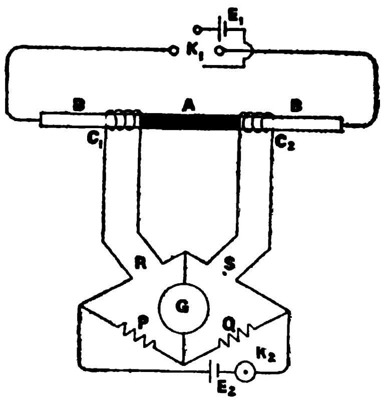


Fig. 3·4

EXPLANATION OF THE EFFECTS : The reason of Peltier effect (and of Seebeck effect as well) lies in the fact that the junction of two dissimilar metals due to transference of electrons become oppositely charged at the interface. As such a junction becomes a seat of *emf* and a potential difference an electric field appears there. The passage of charge caused by a flow of current through the junction calls for a supply or release of energy. In Peltier effect when a current is sent by a cell across any junction in the same direction as the *emf* existing there, energy is to be supplied and as such the seat of *emf* absorbs heat from the junction. Again when the external current flows in the direction opposite to the *emf* of the junction, energy is liberated and the junction gets heated. In a bismuth-antimony couple the *emf* is directed from bismuth to antimony at each of the junctions. Hence an external source driving current from bismuth to antimony causes absorption of heat and the current flowing from antimony to bismuth produces heat. This explains Peltier effect. The

magnitude of the Peltier *emf* depends upon the temperature of the junction. If the two junctions are at unequal temperatures there is a resultant *emf* in the circuit. Hence arises the current due to Seebeck effect.

An equivalent circuit composed of three cells may be considered for conception in the matter. Two cells of *emfs* e_1 and e_2 are in opposition. There will be no current in the circuit if $e_1 = e_2$. If e_1 and e_2 are unequal a current will flow, the direction of the current being determined by relative magnitudes of e_1 and e_2. This simulates the Seebeck effect in a thermo-electric circuit, where unequal temperatures make e_1 and e_2 unequal. Now consider that when e_1 and e_2 are equal a third cell of *emf* E is included in the circuit. This cell sends a current and the cell e_1 which is in series with it joins with it to drive the current spending its own energy. The cell e_2 will be in opposition and heat will be generated in it due to current flowing through it. Here the two cells e_1 and e_2 comprising seats of *emf* may be regarded as representing the Peltier *emfs* at the junctions of a thermo-couple and E is the external source of *emf.*

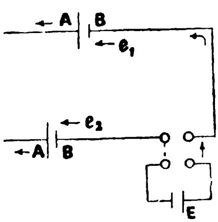


Fig. 3·5

PELTIER COEFFICIENT is defined as the *emf* of at the junction of two dissimilar metals. The energy absorbed (or evolved) at a junction when unit charge passes through it is a measure of this *emf* *i.e.* the Peltier coefficient. It is usually expressed as joules/coulomb *i.e.* as volts and is denoted by π. The amount of heat absorbed (or evolved) in transference of one coulomb of charge if expressed in energy units (joules) gives the magnitude of π in volts.

Magnitude of thermo-emf : Let the two junctions of a thermo-couple be at $T_1°$ and $T_2°$ (Kelvin scale) and the corresponding Peltier coefficients be π_1 and π_2 respectively. If $T_2 > T_1$, the net *emf* in the circuit is $(\pi_2 - \pi_1)$.

The thermo-couple may be regarded as a heat engine pro-

ducing electrical energy by absorbing heat at a temperature and rejecting a portion of it at a lower temperature.

Let us consider that as a result of Seebeck eflect, q coulombs of charge pass round the circuit. The thermal effects at the two junctions are as shown

$$\text{Heat taken at } T_2 = \pi_2 q \text{ joules}$$

$$\text{Heat liberated at } T_1 = \pi_1 q \text{ joules}$$

By applying laws of thermo-dynamics applicable to a reversible heat engine, we may write

$$\frac{\pi_2 q}{\pi_1 q} = \frac{T_2}{T_1}$$

$$\text{or} \quad \frac{\pi_2 - \pi_1}{\pi_1} = \frac{T_2 - T_1}{T_1}$$

$$\text{or} \quad (\pi_2 - \pi_1) = (T_2 - T_1)\frac{\pi_1}{T_1}$$

Since the net *emf* in the circuit $E = (\pi_2 - \pi_1)$

$$\text{so} \quad E \propto (T_2 - T_1)$$

Thus the magnitude of net thermo-*emf* in a circuit should be proportional to the temperature difference of the junctions at a constant temperature of the cold junction. But we find that this does not corroborate the experimental results. So the conclusion becomes inevitable that Peltier *emf* alone cannot explain the behaviour of a thermo-couple completely and there must be some other reversible effect. Actually such an effect exists. This is known as *Thomson effect.*

Thomson effect : It is experimentally found that heat is evolved when a current passes from the hot end to the cold end of an unequally heated copper bar and heat is absorbed if the flow of current is reversed in direction. This suggests the existence of a potential gradient along the length of an unequally heated metal bar. This seems to be logical. Since the pressure of free electrons in a metal depends upon the temperature, a potential gradient should be associated with a temperature gradient. The thermal change in such a case is found to be proportional to the amount of charge transference

and the temperature difference. This phenomenon is known as *Thomson effect.*

Thomson effect is said to be positive (as in copper, zinc, silver) when heat is absorbed when a current flows from cold to hot end and it is negative (as in iron, platinum) where heat absorption is associated with passage of current from hot to cold portion. Heat liberation occurs in the reverse process in the two metals mentioned.

DEMONSTRATION OF THOMSON EFFECT : A long iron rod is bent into U-shape. The bend of the U is heated to redness and the two ends are kept cool in mercury baths at a constant low temperature. Two similar coils P and Q are fixed on the limbs and are connected to the third and fourth arms of a Wheatstone bridge circuit which is kept at balance when there is no current through the rod. A current is passed through the rod using the mercury baths as the terminals. The current passes up and down temperature gradient in the two limbs respectively. The balance of the bridge is found to be upset indicating that the resistances of the two coils (P, Q) have changed. This is due to the liberation of heat in one limb and absorption at the other. The change is resistance is reversed by reversal of current.

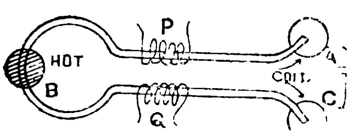


Fig. 3·6

THOMSON COEFFICIENT for a particular material is defined as the heat (expressed in joules) absorbed when a charge of 1 coulomb passes from one point to another 1°C higher in temperature. In such a case the coefficient is positive as in copper. The coefficient is negative if under the said condition there is liberation of heat.

As defined, since Thomson coefficient signifies heat necessary to raise one coulomb of charge to a stage 1°C higher in temperature, it has been named by Kelvin as *specific heat of electricity*, which of course may be negative as well.

Thomson coefficient, denoted by σ, is the *emf* in a conduc-

tor generated due to unit difference in temperature. If σ is the Thomson coefficient in a material the total energy gained or lost in passing of one coulomb of charge from a point at a temperature T_1 to another T_2 should be expressed as $\int_{T_1}^{T_2} \sigma.dT$.

III-3. E. M. F. OF A THERMO-COUPLE

Joint effect of Peltier and Thomson emfs : Consider a couple with junctions at T and $T-\delta T$ temperatures. Let the Peltier coefficients at these temperatures be π and $\pi-\frac{d\pi}{dT}\delta T$. Let σ_a and σ_b be the Thomson coefficients of each member of the couple, both considered as positive. Let a charge of q coulombs circulate round the circuit. The absorption and liberation of energy occur in four stages *viz.* at the temperatures T and $T-\delta T$ at the junctions and through the two metals the ends of which are kept at fixed temperatures.

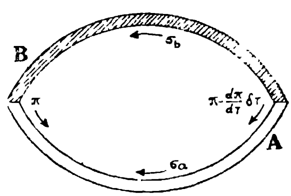


Fig. 3·7

Hence considering energy liberation as negative absorption, the total energy absorption may be expressed as

$$\delta W=\pi q - \left(\pi-\frac{d\pi}{dT}\delta T\right)q-(\sigma_a-\sigma_b)\,\delta T.q \quad \text{joules}$$

$$\text{or} \quad \delta W= q\frac{d\pi}{dT}\,\delta T-(\sigma_a-\sigma_b)\,\delta T.q$$

If δE is the effective *emf* of the couple δW must be equal to $q.\delta E$, hence

$$\delta E = \left[\frac{d\pi}{dT}\,\delta T - (\sigma_a-\sigma_b)\right]\delta T.$$

$$\text{or} \quad Lt.\frac{\delta E}{\delta T}=\frac{dE}{dT}=\frac{d\pi}{dT} - (\sigma_a-\sigma_b)$$

The *emf* of the thermo-couple for two junctions kept at T_2 and T_1 is therefore obtained as

$$\int_1^2 \frac{dE}{dT}dT = \int_1^2 \frac{d\pi}{dT}dT - \int_1^2 (\sigma_a - \sigma_b)\, dT$$

or $$E_{T_1}^{T_2} = (\pi_2 - \pi_1) - \int_1 (\sigma_a - \sigma_b) dT$$

Law of Intermediate Temperatures : $E_{T_1}^{T_2}$ and $E_{T_1}^{T}$ be respectively the *emfs* of a couple between the temperature intervals $T_2 - T$ and $T - T_1$ where $T_2 > T > T_1$.

Hence,

$$E_{T_1}^{T_2} = (\pi_2 - \pi) - \int_T^{T_2} (\sigma_a - \sigma_b) dT$$

and $$E_{T_1}^{T} = (\pi - \pi_1) - \int_{T_1}^{T} (\sigma_a - \sigma_b)\, dT$$

By addition,

$$E_{T}^{T_2} + E_{T_1}^{T} = (\pi_2 - \pi_1) - \int_{T_1}^{T_2} (\sigma_a - \sigma_b) dT = E_{T_1}^{T_2}$$

This equation indicates that the *emf* of a thermo-couple for any temperature interval $(T_2 - T_1)$ is the sum of the *emfs* corresponding to any smaller intervals into which the internal $(T_2 - T_1)$ may be divided. This is the *law of intermediate temperature.* In symbols the law may be expressed as

$$E_{T_1}^{T_p} = E_{T_1}^{T_2} + E_{T_2}^{T_3} + \ldots\ldots + E_{T_{p-1}}^{T_p}$$

Law of Intermediate Metals : Let us consider two separate thermo-couples formed of metals *A-B, B-C.* the junction of each couple being at temperatures T_2 and T_1 where $T_2 > T_1$, (shown in Fig. 3·8), then.

$$E_{ba}=(\pi_{ba})_{T_2}-(\pi_{ba})_{T_1}-\int_{T_1}^{T_2}(\sigma_a-\sigma_b)\,dT \qquad \ldots \text{(i)}$$

$$E_{cb}=(\pi_{cb})_{T_2}-(\pi_{cb})_{T_1}-\int_{T_1}^{T_2}(\sigma_b-\sigma_c)\,dT \qquad \ldots \text{(ii)}$$

By addition,

$$E_{ba}+E_{cb}=(\pi_{ba}+\pi_{cb})_{T_2}-(\pi_{ba}+\pi_{cb})_{T_1}-\int_{T_1}^{T_2}(\sigma_a-\sigma_c)dT \qquad \ldots \text{(iii)}$$

Now consider that three metals A, B, C are joined to form a closed circuit, all junctions being at the same temperature (Fig. 3·8). The resultant *emf* in the circuit is zero since all junctions being at the same temperature there is no source of energy. So we may write

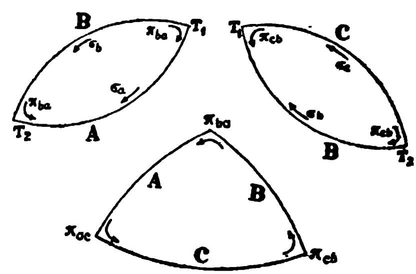


Fig. 3·8

$$\pi_{ba}+\pi_{ac}+\pi_{cb}=0$$

or $\qquad \pi_{ba}+\pi_{cb}=-\pi_{ac}$

or $\qquad \pi_{ba}+\pi_{cb}=\pi_{ca}$

or $\qquad \pi_{bc}-\pi_{ac}=\pi_{ba}$

Substituting these values in (iii) we get,

$$E_{ba}+E_{cb}=(\pi_{ca})_{T_2}-(\pi_{ca})_{T_1}-\int_{T_1}^{T_2}(\sigma_a-\sigma_c)dT=E_{ca}$$

Hence $\qquad E_{ba}+E_{cb}=E_{ca}$

cr $\qquad E_{ba}+E_{cb}+E_{ac}=0$

It follows therefore, $\quad E_{bc}-E_{ac}=E_{ba}$

The last equation shows that the difference of the *emf* of thermo-couples formed by each of the two metals (A, B) with

a third one (C) is equal to the *emf* of the thermo-couple formed by the two (A, B) in question. This equation provides for a method of obtaining the *thermo-emf* of two metals from the values of the *thermo-emfs* of the two when coupled with a third metal. This third metal may be a standard one. Lead has no Thomson effect and as such this is conveniently chosen as this standard. The difference between the two values for the thermo-*emf* of each of the two metals (A, B) with lead as the other member gives the *emf* of the thermo-couple formed by the two (A-B) according to this equation

$$E_{bl} - E_{al} = E_{ba}$$

According to the sign convention, in lead-X couple the *emf* is considered positive when the thermo-current flows from lead to X through the hot junction.

An important fact follows from the result obtained here. Suppose a third metal C is introduced between two metals A-B at one of the junctions, the ends of C joined with A and B respectively being at the same temperature. Let the junction of A-B be at a temperature T_2^{o} while both the junctions of C are at T_1^{o} and $T_2 > T_1$. Then considering all the *emfs* involved (fig. 3·9) we may write

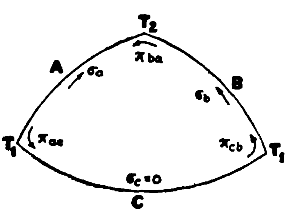


Fig. 3·9

$$[E_{bac}] = [\pi_{ba}]_{T_2} - \int_{T_1}^{T_2} \sigma_a dT + [\pi_{ac}]_{T_1} + [\pi_{cb}]_{T_1} + \int_{T_1}^{T_2} \sigma_b dT$$

or $$[E_{bac}] = [\pi_{ba}]_{T_2} - [\pi_{bc} - \pi_{ac}]_{T_1} - \int_{T_1}^{T_2} (\sigma_a - \sigma_b) dT$$

or $$[E_{bac}] = [\pi_{ba}]_{T_2} - [\pi_{ba}]_{T_1} - \int^{T_2} (\sigma_a - \sigma_b) dT_1 = [E_{ba}]$$

This gives the *Law of intermediate metals*, which may be stated as follows : *the emf of a thermo-couple is unaffected by the introduction of one or more metals kept entirely at the temperature of the circuit at which it is inserted.*

This law has useful applications. When a galvanometer with its leads is inserted in a thermo-electric circuit for the purpose of measurement, it does not affect the *emf* involved, provided that the junctions between the galvanometer and the thermo-couple wires are at the uniform temperature of the point of insertion. Again, the junction of the two metals may be soldered, the material used acting as an intermediate metal does not disturb the *emf* of the particular couple.

III-4. THERMODYNAMICS OF A THERMO-COUPLE

Entropy changes in a circuit : Peltier and Thomson effects are both reversible process and the thermo-couple is a contrivance for conversion of heat energy into electrical energy capable of doing mechanical work. Hence the second law of thermodynamics will be applicable. Although the available energy is small, the thermo-couple may yet be regarded as a heat engine.

Let T_1 and T_2 be the temperature of the junctions in absolute thermodynamics scale where $T_2 > T_1$. Let π_1 and π_2 be the corresponding Peltier coefficients, and σ_a and σ_b are the Thomson coefficients of the two members A and B respectively, both being regarded as positive. At the hot junction the *emf* is directed from B to A. When q coulombs of charge flows completely round the circuit, the heat transference occurs in four steps as detailed below.

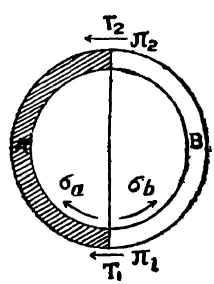


Fig. 3·10

(i) At T_2 heat *absorbed* due to Peltier effect is $\pi_2 q$ joules.

(ii) In passing from T_2 to T_1 through A heat is *liberated* due to Thomson effect. Considering this as negative absorption, heat taken stands as $-\int^{2} \sigma_a dT$ joules.

(iii) At T_1 heat is *liberated* amounting to an absorption of $-\pi_1 q$ joules.

(iv) In passing from T_1 to T_2 through B heat *absorption* due to Thomson effect is $\int_1^2 \sigma_b.dT.$

The process being reversible, the sum of entropy changes may be denoted as $\Sigma\frac{Q}{T}=0$, So,

$$\frac{\pi_2 q}{T_2}-\frac{\pi_1 q}{T_1}-q\int_1^2\frac{(\sigma_a-\sigma_b)}{T}.dT.=0$$

$$\text{or}\quad \frac{\pi_2}{T_2}-\frac{\pi_1}{T_1}-\int_1^2\frac{(\sigma_a-\sigma_b)}{T}.dT=0$$

$$\text{or}\quad \int_1^2\frac{d}{dT}\left(\frac{\pi}{T}\right)dT-\int_1^2\frac{(\sigma_a-\sigma_b)}{T}.dT=0$$

$$\text{or}\quad \frac{d}{dT}\left(\frac{\pi}{T}\right)=\frac{(\sigma_a-\sigma_b)}{T}$$

$$\text{or}\quad (\sigma_a-\sigma_b)=T\frac{d}{dT}\left(\frac{\pi}{T}\right)\qquad\ldots\qquad\ldots\qquad\text{(i)}$$

It has been shown in page 107 that

$$\frac{dE}{dT}=\frac{d\pi}{dT}-(\sigma_a-\sigma_b)$$

Substituting for $(\sigma_a-\sigma_b)$, from (i)

$$\frac{dE}{dT}=\frac{d\pi}{dT}-T\frac{d}{dT}\left(\frac{\pi}{T}\right)$$

$$\text{or}\quad \frac{dE}{dT}=\frac{d\pi}{dT}-\left(\frac{d\pi}{dT}-\frac{\pi}{T}\right)=\frac{\pi}{T}$$

$$\text{Hence}\quad \pi=T\frac{dE}{dT}\qquad\ldots\qquad\ldots\qquad\text{(ii)}$$

Again substituting for $\frac{\pi}{T}$ in (i), we get

$$\sigma_a-\sigma_b=T\frac{d}{dT}\left(\frac{dE}{dT}\right)=\mathrm{T}.\frac{d^2E}{dT^2}$$

If thermo-electric power $\frac{dE}{dT}$ is denoted by P

$$\sigma_a - \sigma_b = T\frac{dP}{dT}$$

If $\sigma_b = 0$, this happens when one of the pair is lead

$$\sigma_a = T\frac{d^2E}{dT^2} = T\frac{dP}{dT} \quad \ldots \quad \ldots \quad \text{(iii)}$$

Equations (ii) and (iii) provide for experimental methods for determination of π and σ. The choice of lead as one of the metals in a thermo-couple is necessary for obtaining σ for a particular metal.

III-5. THERMO-ELECTRIC LINES

Tait's diagram : Thermo-electric power of a thermo-couple is independent of the temperature of the cold junction. If the temperature of the hot junction (T) and the thermo-electric power $P = \frac{dE}{dT}$ be plotted in a graph, a straight line is obtained. Usually such thermo-electric lines for different metals with lead as one member of the thermo-couple are drawn in diagrams. Such a diagram is known as *Tait's diagram*. Some important properties regarding thermo-couple are obtained from a study of such thermo-electric diagrams.

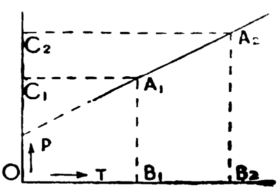


Fig. 3·11

Let us consider the points A_1, A_2 (Fig. 3·11) on a thermo-electric line corresponding to the temperatures T_1 and T_2. Then as shown in the diagram $OB_1 = T_1$, $OB_2 = T_2$, $OC_1 = P_1$, $OC_2 = P_2$.

$$\text{Area} \quad OC_1A_1B_1 = P_1.T_1 = \left(\frac{dE}{dT}\right)_1 . T_1$$

$$\text{Area} \quad OC_2A_2B_2 = P_2.T_2 = \left(\frac{dE}{dT}\right)_2 . T_2$$

Hence Peltier *emf* at $T_1 = \pi_1 = T_1\left(\frac{dE}{dT}\right)_1$ = area $OB_1A_1C_1$

and Peltier *emf* at $T_2 = \pi_2 = T_2 \left(\frac{dE}{dT}\right)_2 =$ area $OB_2A_2C_2$

Again Thomson *emf*, $(\sigma_a - \sigma_b) = T \frac{dP}{dT}$

So Thomson *emf* between $T_2 - T_1 = \int_1^2 (\sigma_a - \sigma_b)\, dT$

$$= \int_1^2 T \frac{dP}{dT}.dT = \int_1 T.dP$$

$$= \text{area } A_1C_1C_2A_2$$

Hence the *emf* of a thermo-couple obtained as

$E = (\pi_2 - \pi_1) - \int_1^2 (\sigma_a - \sigma_b) dT$, may be written as

$E =$ area $OB_2A_2C_2 -$ area $OB_1A_1C_1 -$ area $A_1C_1C_2A_2$

or $E =$ area $A_1B_2B_2A_2 = (T_2 - T_1)\left(\frac{P_2 + P_1}{2}\right)$

Thus the *emf of a thermo-couple is equal to the product of the average thermo-electric power of the junctions and the temperature difference between them.*

Thomson coefficient for lead is zero and so if circuit is made with lead as one of the members of the couple we have in the expression, $(\sigma_a - \sigma_b) = T \frac{dP}{dT}$, $\sigma_b = 0$ and hence

$\sigma_a = T \frac{dP}{dT}$. Thermo-electric diagrams for different metals are usually drawn with lead as one of the members. If the thermo-electric line for such a pair slopes downwards then σ is considered negative for the metal and it is positive when the line slopes upwards.

Let a thermo-couple be made with two metals A and B whose thermo-electric line with respect to lead as the other metal are drawn respectively as $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. As shown in the diagram σ is positive for A and negative for B. Let a couple be formed with the metals A and B

having the junctions at T_1 and T_2. In the diagram $OC_1 = T_1$, and $OC_2 = T_2$.

From the diagram,

$$[\pi_{l a}]_{T_1} = T_1.A_1C_1$$

$$[\pi_{l b}]_{T_1} = T_1.B_1C_1$$

$$[\pi_{ba}]_{T_1} = [\pi_{l b} - \pi_{l a}]_{T_1}$$

$$= T_1[B_1C_1 - A_1C_1]$$

$$= T_1.A_1B_1$$

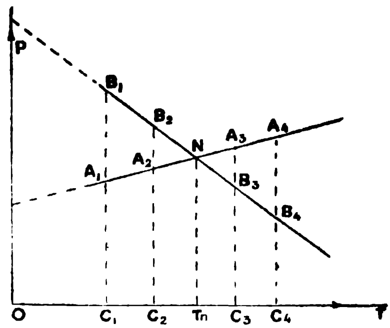


Fig. 3·12

Since $[\pi_{ba}]T_1 = T_1 \times$ thermo-electric power at T_1, so comparing this with the relation obtained in the foregoing steps, we find that

$$A_1B_1 = \text{thermo-electric power of } (A\text{-}B) \text{ couple.}$$

Therefore it is to be concluded that the *thermo-electric power of two metals when they form the couple is the difference of the thermo-electric powers of each of the metals with respect to a third metal.*

Also, by law of intermediate metal, the *emf* of A-B couple is obtained as

$$E_{ab} = E_{lb} - E_{la}$$

so $E_{ab} = \text{area } B_1B_2C_2C_1 - \text{area } A_1A_2C_2C_1$

$$E_{ab} = \text{area } B_1B_2A_2A_1$$

If now maintaining the temperature of the cold junction unchanged, the temperature of the hot junction is gradually raised the area $B_1B_2A_2A_1$ increases showing that the *emf* rises with temperature. This goes on until the point N, the point of intersection of the two lines is reached. Beyond this the resultant *emf* diminishes with rise of temperature of the hot junction. For a temperature T_3 represented by OC_3 beyond N, the *emf* is the difference of the two areas on the two sides of N, *i.e.* between the areas B_1NA_1 and A_3NB_3.

The temperature T_n corresponding to the point N is the *neutral temperature*. It is seen that the thermo-electric power at this temperature is zero. Since at the neutral temperature

the thermo-electric power $P=\frac{dE}{dT}=0$, at this temperature the *emf* is maximum.

It is clear from the diagram that if the temperatures of the hot and cold junctions are equally removed from the neutral temperature and are on the opposite sides of it, the *emf* in the circuit is zero. The temperature of the hot junction corresponding to this is the *inversion temperature* and for any temperature of the hot junction higher than the inversion temperature the *emf* is reversed.

III-6. THERMO-ELECTRIC CONSTANTS

E.M.F. **Equation** : From data obtained from experiments it has been possible to express the *emf* of a thermo-couple in terms of the terms of the temperature (t°C) of the hot junction when the cold junction is at the temperature of melting ice (0°C). In such a case the *emf* is obtained as

$$E = \alpha t+\tfrac{1}{2}\beta t^2$$

α, β are called the thermo-electric constants.

If the cold junction be not at 0°C, *but the junctions are at temperatures* t_2 and t_1 ($t_2>t_1$), then the *emf should be obtained according to the law of intermediate temperatures.*

$$E=\alpha(t_2-t_2)+\tfrac{1}{2}\beta(t_2^2-t_1^2)$$

We have $E=\alpha t+\frac{1}{2}\beta t^2$

Thermo-electric Power $\frac{dE}{dT}=\frac{dE}{dt}=\alpha+\beta t$

If the junctions are at T_1°K and T_2°K, $\frac{dE}{dt}=\alpha+\beta(T_2-T_1)$

Peltier Coefficient at T_2°K, $\pi_2=T_2\frac{dE}{dT}$

Hence, $\pi_2=\alpha T_2+\beta T_2(T_2-T_1)$

or $\pi_2=(\alpha-\beta T_1)T_2+\beta T_2^2$

Thomson Coefficient, $(\sigma_a - \sigma_b) = T\dfrac{d^2E}{dT^2}$

Since $\dfrac{dE}{dT} = \alpha + \beta t$, $\dfrac{d^2E}{dT^2} = \beta$.

Hence $\sigma_a - \sigma_b = \beta.T$.

Neutral temperature : Since at neutral temperature $\dfrac{dE}{dt} = 0$ and again $\dfrac{dE}{dt} = \alpha + \beta t$, neutral temperature is obtained from the equation $\alpha + \beta(T_n - T_1) = 0$, where T_n and T_1 an respectively the neutral temperature and the cold junction temperature in the Kelvin scale. If $T_1{}^\circ K = 0^\circ C$, $T_n - T_1 = t^\circ{}_n C$. In such a case putting $\alpha + \beta.t_n = 0$, we get neutral temperature $t_n = -\dfrac{\alpha}{\beta}$ in degrees celsius (centigrade).

'*No emf*' condition is obtained by putting $E = 0$ in the *emf* equation $E = \alpha t + \frac{1}{2}\beta t^2$. This gives the values of t as $t = 0$ and $t = -2\alpha/\beta$. The first value indicates that the two junctions are at the same temperature and the second value refers to the inversion temperature in degrees celsius (centigrade).

The thermo-electric constants of a few common metals with reference to lead as the other member of the couple are given in the following table. The constants applied to the formula $E = \alpha t + \frac{1}{2}\beta t^2$ give the *emf* in *microvolts* of the thermocouple of the metal concerned and lead.

TABLE OF CONSTANTS

METALS	α	β
Antimony	+35·6	+0·145
Bismuth	−74·42	+0·032
Constantan	−37·76	−0·079
Copper	+2·76	+0·012
Iron (soft)	+16·65	−0·030
Nickel	−16·30	−0·027
Silver	+3·34	+0·008
Zinc	+3·10	−0·032

ILLUSTRATIVE EXAMPLES

1. *One end of an iron-lead couple is maintained at 0°C and the other at 100°C. Thermo-electric power of the couple is 13·5 μV per degree celsius at 0°C and it is —10·3 μV per degree celsius at 100°C.*

Calculate (a) the Peltier coefficients, (b) the Thomson *emf* and (c) the *total emf*.

Solution : (a) Peltier coefficient $\pi = T\frac{dE}{dT}$

Hence $\pi_{100} = 373\,(-10{\cdot}3) = -3841{\cdot}9\ \mu V$

$\pi_0 = 273\,(13{\cdot}5) = 3685{\cdot}5\ \mu V$

(b) Thomson coefficient $\sigma = T\frac{d^2E}{dT^2}$

$$\frac{d^2E}{dT^2} = \frac{dP}{dT} = \frac{P_{100} - P_0}{100} = \frac{-10{\cdot}3 - 13{\cdot}5}{100} = -0{\cdot}238\ \mu V$$

$$e = \int_{273}^{373} \sigma.dT = \frac{d^2E}{dT^2}\int_{273}^{373} T.dT = -0{\cdot}238\left[\frac{T^2}{2}\right]_{273}^{373} = -11{\cdot}9 \times 646$$

Hence $e = -7687{\cdot}4$ micro-volts.

(c) $E =$ Average $\frac{dE}{dT} \times$ temperature difference

$$= \frac{13{\cdot}5 + (-10{\cdot}3)}{2} \times 100 = 1{\cdot}6 \times 100 = 160\ \mu V$$

Alternatively, $E = \pi_{100} - \pi_0 - \int_{273}^{373} \sigma dT$

or $E = -3841{\cdot}9 - 3685{\cdot}5 - (-7687{\cdot}4)$

or $E = 160\ \mu V$.

2. *Thermo-electric powers of silver and iron coupled with lead are respectively obtained from the relations 3·34+0·008 t and 16·65—0·0:0 t expressed in microvolts per 0°C, t being the temperature in degree celsius. Determine the emf of silver-iron couple with junctions at 20°C and 100°C.*

Solution : The thermo-electric power of silver-iron couple

are obtained by taking the difference of thermo-electric power with lead. So,

$$P_{20}=(16{\cdot}65-0{\cdot}030\times 20)-(3{\cdot}34+0{\cdot}008\times 20)=12{\cdot}55$$

$$P_{100}=(16{\cdot}65-0{\cdot}030\times 100)-(3{\cdot}34+0{\cdot}008\times 100)=9{\cdot}510$$

$$\text{Average } P=\tfrac{1}{2}(12{\cdot}55+9{\cdot}510)=11{\cdot}03$$

$$E.M.F=11{\cdot}030\times 80=882{\cdot}40\ \mu V.$$

3. *Calculate with the help of the data given in the foregoing table of constants, the neutral temperature for iron-copper couple.*

Solution : We have $E_{il}-E_{cl}=E_{ic}$, hence

$$(\alpha_i t+\tfrac{1}{2}B_i t^2)-(\alpha_c t+\tfrac{1}{2}\beta_c t^2)=\alpha_{ic}t+\tfrac{1}{2}\beta_{ic}t^2$$

or $$\alpha_{ic}t+\tfrac{1}{2}\beta_{ic}t^2=(\alpha_i-\alpha_c)t+\tfrac{1}{2}(\beta_i-\beta_c)t^2$$

So for the iron-copper couple $\alpha_{ic}=\alpha_i-\alpha_c$, $\beta_{ic}=\beta_i-\beta_c$. Substituting for values of α_i, α_c, β_i and β_c from table

$$\alpha_{ic}=16{\cdot}65-2{\cdot}76=13{\cdot}89$$

$$\beta_{ic}=-0{\cdot}030-0{\cdot}012=-0{\cdot}042$$

Neutral temperature $= t_n=-\dfrac{\alpha_{ic}}{\beta_{ic}}=\dfrac{13{\cdot}89}{{\cdot}042}=331°C.$

4. *Determine the Peltier coefficients and Thomson emf of a couple whose thermo-emf is obtained as $E=19{\cdot}64t+0{\cdot}035t^2$ μV, t being the temperature difference in degree celsius of the junctions kept at 0°C and 100°C.*

Solution : $E = 19{\cdot}64t+0{\cdot}035t^2$

$$\frac{dE}{dt}= 19{\cdot}64+0{\cdot}070t,\quad \frac{d^2E}{dt^2}=0{\cdot}070$$

$$\pi= T\frac{dE}{dT},\quad \text{hence } \pi_0=273\times 19{\cdot}64=5{\cdot}36\ mV.$$

$$\text{and}\quad \pi_{100}=373(19{\cdot}64+7{\cdot}00)=9{\cdot}93\ mV$$

$$\text{and}\quad \sigma_1-\sigma_2=T\frac{d^2E}{dT^2}=0{\cdot}070T.$$

III-7. THERMO-ELECTRIC MEASUREMENTS

Measurement of Peltier Coefficient : The junctions of the thermo-couple ($A-B-A$) for which the Peltier coefficient is

to be determined are inserted in two Dewar vessels (D_1, D_2 in Fig. 3·13). The vessels contain a non-conducting liquid in quantities which would make the thermal capacities of the two vessels with their contents equal. Any difference in temperature of the two vessels is to be detected by a copper-constant thermoco-uple (T) in series with the galvanometer (G). A battery (E_1) sends a current through the junctions of the thermo-couple under observation. Due to Peltier effect one of the junction will be heated and the other cooled. But Joule's heating effect will cause heat production in both proportional to the resistances R_1 and R_2 of the conductors inside. To equalise the rise in temperature in the two vessels due to all the effects resistances r_1 and r_2 are introduced in the two vessels respectively. By passing a suitable current in one of the two at a time from a separate battery (E_2) circuit heat produced in the two vessels are equalised which may be ascertained by obtaining a balance in the galvanometer in the copper-constant thermo-electric circuit.

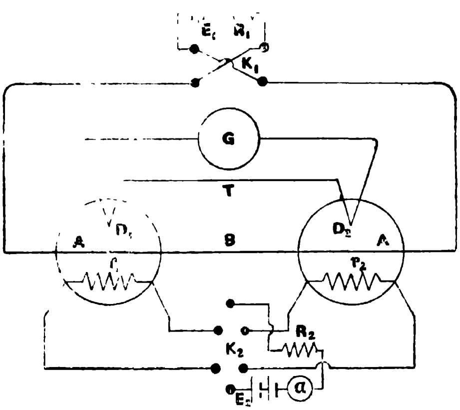


Fig. 3·13

Let π be the Peltier coefficient and i be the current through the thermo-couple under observation, i_2 the current in the resistance r_2. The heats generated in the two vessels under the arrangement are

$$\pi i + i^2 R_1 \quad \text{and} \quad -\pi i + i^2 R_2 + i_2^2 r_2$$

and if these are the values after adjustment to equality then

$$i^2(R_1 - R_2) = 2\pi i - i_2^2 r_2 \qquad \text{(i)}$$

The current in the thermo-couple (under investigation) circuit is now reversed and this time for balance in the copper-constant circuit current i_1 is passed through r_1 instead of r_2, which however carries no current. Now for a balance this time

$$i^2(R_1-R_2)=-2\pi i+i_2^2r_2 \qquad \text{(ii)}$$

From (i) and (ii)

$$\pi=\frac{i_1{}^2r_1+i_2{}^2r_2}{4i}$$

i_2 and i_2 are measured by means of an ammeter, r_1 and r_2 are known resistances. π is calculated from the above equation. This method is due to Caswell.

Measurement of thermo-emf : In a simple way the *emf* of a thermo-couple may be determined by means of a potentiometer. Let a potentiometer (Fig. 3·14a) wire having a resistance r include in its circuit an external resistance R and a cell of *emf* E. If the current in the circuit be i the potential drop per unit length of the potentiometer wire of length L is given by ir/L. Putting the value of i, this potential drop is obtained as $\frac{E}{R_1+r}\cdot\frac{r}{L}$. If a thermo-couple is balanced at a length l on the wire, its *emf* is given by $\frac{Er}{R_1+r}\cdot\frac{l}{L}$.

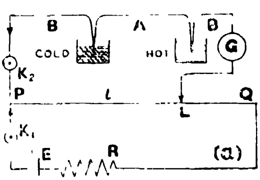


Fig. 3·14a

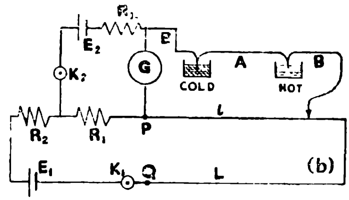


Fig. 3·14b

This method has some drawbacks. For accurate determination a modified a arrangement using a standard cell is preferred. The circuit diagram is as shown (Fig. 3·14b). R_1 and R_2 are two resistances of such values that the potential drop along the potentiometer is of the order of the themo-*emf*. E_2 is a standard cell with a high resistance R_x in series.

The values of R_1 and R_2 are so adjusted that there is no current in the galvanometer (G) on closing the keys K_1 and K_2. When this is obtained the potential drop across R_1 is equal to the *emf* (e) of the standard cell. So the current in the resistance R_1 as well as in the potentiometer wire is e/R_1. The potential drop per unit length of the wire of length L and resistance r is given by $\frac{e}{R_1} \cdot \frac{r}{L}$. If the thermo-couple *emf* E be balanced at a length l then $E=\frac{er}{R_1} \cdot \frac{l}{L}$.

One junction of the thermo-couple is put in melting ice, the other junction is heated to different known temperatures and the *emf* is balanced for each temperature and calculated from the expression for E. While the thermo-couple is put in circuit the standard cell should be disconnected by opening the key K_2.

$\frac{dE}{dT}$, *i.e.*, the thermo-electric power for any temperature may be obtained from the slope of the E-t curve.

Determination of constants α *and* β. The equation representing the relation between the *emf* and temperature difference of the junctions is given by

$$E=\alpha t+\tfrac{1}{2}\beta t^2$$

$$\text{or} \quad \frac{E}{t}=\alpha+\tfrac{1}{2}\beta t$$

A series of values of E corresponding to different temperatures of the hot junction (t), the other junction being at $0^\circ C$, is obtained. E/t is plotted against t. The slope of the straight line graph obtained gives $\frac{1}{2}\beta$ and its intercept on the E/t axis is a measure α.

Determination of Thomson coefficient : Since $\sigma_a-\sigma_b=\beta.T$ (*See ch. III*-6) a knowledge of β determines $\sigma_a-\sigma_b$. If the material be lead $\sigma_b=0$, we have $\sigma_a=\beta T$.

III-8. APPLICATIONS OF THERMAL EFFECTS

Hot wire Instruments : Heating effect of current is utilised to measure current in a hot wire ammeter. The essential part

of the instrument is a fine platinum-irridium wire (*R-R* in Fig. 3·15) kept stretched with ends connected to the terminals of the instrument. A fibre connected at the middle of the wire passes round a grooved pulley (*W*) and its other end is attached to a fixed spring (*S*) which keeps the system taut. A light pointer (*P*) attached to the pulley moves over a scale when the pulley rotates.

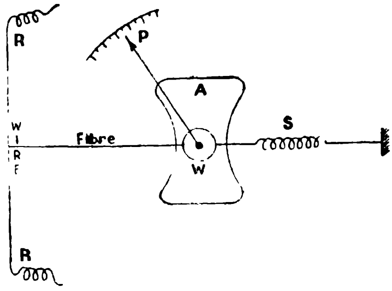


Fig. 3 15

If a current flows in *R-R* the wire is heated and on expansion it sags. The slackening of the tension is taken up by the spring through the fibre and the pulley rotates causing the pointer *P* to move over the calibrated scale. The increase in the length of the wire depends upon the amount of heat produced which again is proportional to the square of the current. The scale readings are obtained by passing known direct currents.

The oscillations are damped by the movement of an aluminium vane (*A*) attached to the pulley in a field due to a horse-shoe magnet.

When used as a voltmeter the scale is to be calibrated in volts.

Current carrying capacity of this type of instruments is limited as the wire is fine. No shunt is necessary for currents upto an ampere. When used as a voltmeter a series resistance may be necessary. The instruments are sluggish in action. These are suitable for alternating current and voltage, since the heating effect is proportional to the square of the current.

Thermo-galvanometer : This is a current measuring device based' on thermo-electric and heating effects. A loop of silver wire whose ends are attached to two bars, one of antimony and the other of bismuth respectively hangs between the pole-pieces

of a magnet and the tips of the bar are soldered. Just below the junction there is a resistance wire through which the current to be measured is passed. The wire gets heated and the radiation from it causes a thermo-electric current in the thermo-couple circuit of which the loop is a part. The deflection of the loop is a measure of the current. High frequency currents of a few micro-amperes may be measured with this instrument.

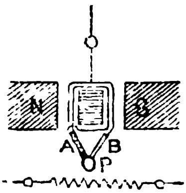


Fig. 3·16

Thermopile, Radio micrometer, thermo-couple pyrometer are instruments based on thermo-electric effects for measurement of radiant heat and high temperature. These come within the purview of Heat and as such are not discussed in this treatise.

EXERCISES ON CHAPTER III

3-1. Obtain the relation between the heat developed in a conductor and the current flowing through it.

Show that in branching of current in a network the distribution is such that heat production is minimum.

Obtain the condition for maximum heat production in a circuit drawing a current from battery.

3-2. Describe in brief the thermo-electric effects in a circuit. Discuss the theoretical reasoning which led Lord Kelvin to conclude that the Peltier effect alone cannot explain the behaviour of a thermo-couple.

3-3. Discuss the thermodynamics of a thermo-couple and obtain the values of Peltier and Thomson coefficients in terms of quantities which can be experimentally determined.

3-4. Deduce an expression for the thermo-couple *emf* in terms of the Peltier and Thomson coefficients. Deduce and explain the laws of intermediate temperatures and intermediate metals.

3-5. How can you demonstrate the existence of Peltier and Thomson effects ?

Describe an experimental method of determining Peltier coefficient of a thermo-couple.

3-6. Explain the terms : Thermo-electric line, Neutral temperature, Inversion temperature, Thermo-electric power.

What is Tait's diagram ? Discuss how from the diagram you can obtain the values of Total *emf*, Peltier and Thomson cocfficients.

3-7. Define Thermo-electric power. Prove the relations

$$\pi = T\frac{dE}{dT} \text{ and } \sigma_a - \sigma_b = T\frac{d^2E}{dT^2}$$

3-8. Give the equation representing the thermo-*emf* in terms of constants and temperature difference of the junctions. Obtain the different quantites involved in terms of these constants. How can these constants be determined experimentally ?

3-9. What is thermo-electric diagram ? Discuss the importance and indicate the informations which can be obtained from it.

In an experiment with a thermo-couple having one junction at $0°C$ it was found that for hot junction kept at $10°C$ and $30°C$ the *emfs* are respectively $98\mu V$ and $202\mu V$. Calculate the neutral temperature and thermo-electric power at $100°C$.

[*Ans* : $250°C$, $5{\cdot}8\ \mu V$ per $°C$]

3-10. Given that the thermo-electric power of lead-copper couple may be expressed as $136+0{\cdot}95t\ \mu V$ per $°C$ and that of lead-iron as $1734-4{\cdot}87t\ \mu V$ per $°C$, where t is in degrees. Calculate the *emf* of copper-iron couple when the junctions are at $1°C$ and $100°C$.

[*Ans* : 0·13 volt]

3-11. What is Tait's diagram ? Prove that the *emf* of a pair of metals may be obtained as the product of average thermo-electric power between the temperature of the cold and the hot junctions and the temperature difference.

3-12. Show how you can obtain the *emf* of a thermo-couple of two metals from the thermo-electric lines of the

thermo-couple of the metals separately formed with a standard metal.

The thermo-electric power of lead-copper couple is

$$2{\cdot}76+0{\cdot}12\ \mu V \text{ per degree celsius and}$$

for a lead-constant couple it is

$-37{\cdot}76-0{\cdot}079t\ \mu V$ per degree celsius. Calculate the *emf* in a copper-constant couple whose junctions are at $0°C$ and $200°C$ respectively. Obtain the neutral temperature.

[*Ans*: $8286\ \mu V$; $445{\cdot}2°C$]

3-13. The thermo-emf in a thermo-couple is given by

$$E=13{\cdot}31t-0{\cdot}022t^2\ \mu V$$

where t is the temperature difference in degress celsius of the junctions, the cold junction being at $0°C$. If the hot junction temperature is $100°C$, calculate Peltier coefficients of the junctions and the Thomson *emf* of the couple.

[*Ans*: $\pi_{100}=3{\cdot}32\ mV$, $\pi_0=3{\cdot}63\ mV$, $\sigma_a-\sigma_b=0{\cdot}044T$]

3-14. Describe the experimental method of accurately determining the *emf* of a thermo-couple. State how you can hence obtain the thermo-electric power and thermo-electric constants.

3-15. What is a Hot-wire ammeter ? Describe its working principle and discuss its merits and demerits.

CHAPTER IV

ELECTROLYTIC CONDUCTION

IV-1. MECHANISM OF CONDUCTION IN SOLUTIONS

Ionic Dissociation : When an electric current is passed through a metallic conductor, no change is produced at the terminals through which the current enters and leaves or in any part of the conductor. This is otherwise in case of conduction through solutions. The passage of current through a solution causes a deposit of material substances on the electrodes *i.e.* the terminals. The deposit on the two electrodes are not similar. Further, amount of deposit is found to be proportional to the strength of the current and also to the time through which it flows. These facts were first enunciated by Faraday as laws of electrolysis. Pure liquids (except mercury, which is a metal) do not conduct electricity. The reason for this is attributed to the fact that they contain no free electrons. But it is found that liquids having some dissolved chemical compounds are in many cases found to become conductor. Pure water is a very poor conductor of electricity but it is observed that a small quantity of dissolved chemical salt increases the conductivity. Therefore it must be assumed that carriers of current in some form or other are formed in solutions.

ARHENIUS' THEORY : In order to explain the conduction of current through a solution Arhenius introduced the theory of electrolytic dissociation which asserts that a liquid having some particular kinds of chemical compounds in solution contains charged particles, both positive and negative. These charged particles are called *ions* and these are produced by the spontaneous disintegration of the molecules of the compound in solution in two parts carrying equal quantity of opposite charges. Thus in the case of $AgNO_3$ molecules which on being dissolved in water splits upto a certain fraction of the

total number of molecules and free Ag^+ and NO_3^- ions are formed. These ions move at random in all directions inside the solution. The solution indicates no net charge of any kind, there being equal number of positive and negative ions. There is also no net movement of charge of one kind in any particular direction, the motion of two kinds of ions, carrying opposite charges being random.

When the ions in the solution come under a directive influence as may be done by establishing a potential gradient inside the solution, the positive and negative ions are drifted in mutually opposite directions towards the negative and positive electrodes respectively. This constitutes a current flowing through the solution. The current inside the solution is carried by the molecules or groups of molecules in the form of ions. As such deposit and reactions are expected at the electrodes where the ions terminate their journey through the solution. The terminal which receive the positive ions, known as *cations*, is called the *cathode* and the negative ions or *anions* are the particles which migrate upto the terminal called the *anode*. Metallic ions are deposited on cathode and they carry positive charge and the non-metallic ions reaching the anode are negatively charged.

Since according to Arhenius the current through a solution is entirely due to the motion of ions, the conductivity of solution should depend upon the number of free ions present. It is found that any potential difference, however small, causes a current to flow. Hence it becomes evident that a solution contains at least a few ions, otherwise a minimum potential difference would have been necessary to effect the separation of particles carrying opposite charges forming a molecule.

Factors causing dissociation : Any agency which weakens the attractive force between the opposite charges in the molecule will tend to dissociate the molecules. The force between opposite charges depends on permittivity (dielectric constant) of the medium in which these are placed. Since liquids possess high permittivity, the bond between opposite charges become weak in solutions and the impact of molecules

due to thermal agitation causes some molecules to split up into ions. Clausius pointed out that the equilibrium condition in solution (electrolyte) cannot be such that ions of opposite charges remain firmly bound together. The equilibrium is a dynamic one so that ions are continuously changing partners and a number of ions are momentarily present as free ions. During such free state an applied electric field in the solution directs these in appropriate directions and such flow of oppositely charged ions constitutes the electric current.

Reactions at the electrodes: When the ions arrive at the electrodes their charges are neutralised. The atoms or groups of atoms are then either deposited on the electrodes or liberated there, otherwise they may as well be involved in chemical reactions at the electrodes. An illustrative example is discussed.

When an electric current is passed through $CuSO_4$ solution Cu^{++} ions are directed towards the cathode and at the end of journey they give up their charges and Cu atoms are deposited on the cathode. The SO_4^{--} ions move towards the anode and lose their charges on reaching there. The uncharged SO_4 radical cannot remain as such and the final result depends upon the nature of electrode. If the electrode be copper, SO_4 combines with Cu to from $CuSO_4$ which again goes into solution. As a result the total quantity of $CuSO_4$ in solution, in other words the concentration of the solution remains unaltered. The net result of the current flow being a transference of copper from anode plate to the cathode. If the anode is made of platinum SO_4 radical reacts with H_2O and oxygen is liberated at the anode.

$$2SO_4+2H_2O=2H_2SO_4+O_2$$

EVIDENCE FROM OSMOTIC PRESSURE: It follows from the laws of osmosis that equimolecular solution (having the same number of molecules per litre) of different substances in water have the same osmotic pressure and hence the same freezing point. It is found that osmotic

pressure (obtained by freezing point determination) of electrolytic solutions is greater than that of equimolecular solution of 'normal' substances, such as cane sugar, which in solution do not form electrolyte. In all cases electrolytic solution behave as if they contain abnormally large number of molecules. This is explained by assuming that the ions in solution each behave as separate solute molecule. It thus confirms the idea of electrolytic dissociation. According to Arhenius the production of extra particles by dissociation is responsible for the abnormal rise of osmotic pressure in case of electrolytic solutions.

An objection to Arhenius' theory was that it would be impossible for elements such as sodium which reacts with water to exist as ions in water or for substances, as chlorine, which are soluble in water to remain as free ions instead of initiating the bleaching action in dissolved state. Such objections are met with if it is remembered that an ion is in an entirely different condition from that of the uncharged atom. The chemical properties of an atom is attributed to the number of electrons in the outer shell. A sodium atom which has only one electron in the outer shell loses it when it becomes an ion and having ten (2+8) electrons in the two saturated shells behaves as inert neon atom. In fact it is found that chemical reactive properties is restored when the ions give up their charges to the electrodes.

Effect of temperature : Unlike metallic conduction electrolytic conduction is accompanied by transport of atoms and molecules. Metallic conductivity, which is due to free electrons, decreases with rise of temperature. This is because of the fact that retardation in motion of electrons is increased due to increased thermal agitation of molecules. But is found that rise of temperature causes better conduction in electrolytes. This is attributed to the dimunition of viscosity of water, reducing the resistance offered to the passage of ions. Ionic mobility and hence the conductivity increases by about two percent for each celsius degree of temperature rise.

IV-2. PHENOMENON OF ELECTROLYSIS

Faraday's Laws : Faraday experimentally found that when an electric current is passed through an electrolytic solution there is splitting of dissolved chemical compound in its constituent parts. This is known as electrolysis. There are deposits or liberations of substances at the electrodes. Faraday obtained a relation between the quantity of eleclectricity passed through the electrolyte and the mass of substance liberated.

FIRST LAW : *The mass of any substance liberated by electrolysis is directly proportional to the quantity of charge that has passed through the electrolyte.*

SECOND LAW : *The masses of different substances liberated during electrolysis by the passage of same quantity of charge are proportional to their respective chemical equivalents.*

Chemical equivalent is in general obtained as the ratio, atomic weight/valency of the element. If expressed as *gramme-equivalent* it is the mass in grammes which in a chemical reaction combines with or displaces eight grammes of oxygen.

Electro-chemical equivalent : The first law as stated in the foregoing lines may be expressed as

$$m \propto \int^{t} i.dt$$

where m is the mass of deposit in time t by a current i causing a passage of charge q. The above relation may be written as $m = Zit = Zq$.

Z is a constant, called the *electro-chemical equivalent* of the element. *It is defined as the mass of ion deposited by one coulomb of charge or by one ampere current in one second.*

Faraday : It is found that the gramme-equivalent of a substance divided by its electro-chemical equivalent *i.e. the amount of charge required to liberate one gramme-equivalent is the same for all elements.* Taking for instance, the case of silver of which the gramme-equivalent is 107·88 and electro-

chemical equivalent is 0·0011180, the quantity of charge required for liberation of one gramme-equivalent of silver is obtained as

$$F=\frac{107{\cdot}880}{0{\cdot}001118}=96493{\cdot}7=96500 \text{ (approx.)}$$

If Z for silver is taken as 0·0011183 F becomes equal to 96470. This amount of charge expressed in coulombs is called a ***Faraday***.

Ionic charge : Since the gramme-equivalent of all monovalent atoms contain the same number of atoms it is to be concluded from the definition of faraday that the liberation of each monovalent atom from the electrolyte requires the of same amount of charge. It confirms the idea that the charged atoms are carriers of electricity and all monovalent ions carry a constant and fixed amount of charge, whatever be its chemical nature and a divalent ion carry the double amount of charge.

Each gramme-atom of an element contains ths same number of atoms denoted by N, the *Avogadro number*. For a monovalent atom, gram-atom is same as the gramme-equvalent as in the case of silver. N ions of silver carry one faraday of charge, hence the charge associated with each ion of silver is

$$e=\frac{F}{N}=\frac{96470}{6{\cdot}023\times10^{23}}=1{\cdot}601\times10^{-19} \text{ coulomb.}$$

This is the minimum amount of charge to be obtained from an ion and so this must be the charge carried by a single electron or proton. We cannot get any charge less than this. Hence comes the idea of *atomicity of electricity*. It is the natural unit of electric charge and in terms of our conventional unit of charge, one electronic charge may be expressed as $1{\cdot}601\times10^{-20}$ e. m. units or $4{\cdot}803\times10^{-10}$ e. s. units of charge. Alternatively, electrostatic unit charge may be defined as the charge carried by 2082×10^{6} electrons.

International Ampere : Faraday's law and the idea of elctrochemical equivalent provides for a convenient working definition of ampere, the practical unit of current. It is

measured as the constant current whitch when passed through an aqueous solution of silver nitrate causes a deposit of silver at the rate of 0·0011180 gms. per second. This unit is called *international ampere.* Consequently *international coulomb* is the amount of charge depositing 0·0011180 gm. of silver from a silver nitrate solution.

Polarisation in electrolysis : It is found that any potential difference however small causes a current to flow through an electrolyte, but the current is stopped very soon unless the potential difference between the electrodes excceds a minimum value. In electrolysis of acidulated water this potential difference is about 1·7 volts. The reason for stopping of the current is attributed to the deposit of ions on the electrode forming a layer of charge which produces an opposing *emf.* This is called *polarisation.* This may be observed by arranging electrolysis of water between two platinum electrodes.

The experimental arrangement is as shown in sketch (Fig. 4·1). The liberated oxygen and hydrogen are respectively deposited on the anode and the cathode of the water voltameter (V). Very soon there is a layer of hydrogen ions on the cathode and oxygen lies on the anode. This causes an opposing *emf* directed from cathode to anode inside the voltameter. This is the back *emf* tending to send a current in the opposite direction and very soon electrolysis is stopped. If now the battery be disconnected and a circuit is completed externally through a galvanometer (G) the galvanometer shows a momentary deflection indicating the effect of back *emf.* Therefore for continuous electrolysis of water the *emf* of the battery must exceed the back *emf* created in such a case. The back *emf* is found to be 1·7 volts. A single Daniell cell which has an *emf* of 1·1 volts cannot be applied to decompose water continuously. A minimum vollage is necessary to overcome the effect of polarisation.

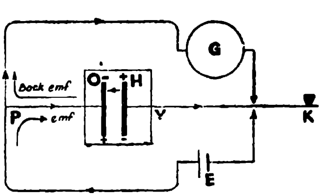


Fig. 4·1

Internal resistance of electrolyte : When the current passed through an electrolyte is small and lasts only for such a short time that there is no change in concentration, the relation between the applied *emf* (E) and the current (i) produced is found to be linear but the straight line does not pass through the origin (Fig. 4·2). It is due to back *emf*. If E is the applied *emf* and e is the back *emf*, the resistance of the liquid is given by $R=(E-e)/i$. Where there is no polarisation, as in the case of electrolysis of copper sulphate solution with copper electrodes, the internal resistance should be calculated as E/i.

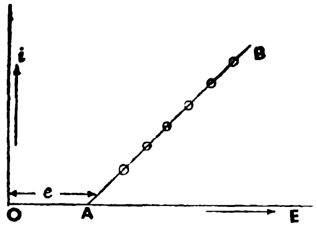


Fig. 4·2

If in an electrolysis E is the applied *emf*, e is the back *emf* and R is the resistance of the electrolyte, the current i is given by

$$i=\frac{E-e}{R}$$

$$\text{or} \quad Ei=Ri^2+ei$$

The power supplied is Ei and it is spent in two parts, Ri^2 is the power consumed in heat production and ei represents the power used up in chemical changes.

IV-3. CONDUCTIVITY AND CONCENTRATION

A few definitions : There are some terms used in connection with conduction through solutions.

CONCENTRATION : The number of gramme-equivalents of the solute in one litre of solution is measured as the *concentration* (c) of the solution.

DILUTION : It is the volume in litres which contains one gramme-equivatent of the solute. If it is denoted by d, by definition $c=1/d$.

SPECIFIC CONDUCTIVITY OR CONDUCTIVITY : It is the reciprocal of resistance of a column of liquid one centi-

metre long and one square centimetre in cross-section. It is denoted by K.

EQUIVALENT CONDUCTIVITY : It is the conductivity of that quantity of solution which contains one gramme-equivalent of the solute when measured with electrodes one centimetre apart. If it be denoted by λ, then $\lambda = Kd = K/c$.

The importance of the idea of equivalent conductivity is that λ is a measure of the conducting power of a fixed quantity of the solute irrespective of dilution. The quantity, as defined, should not vary with change of concentration. But this not true in all cases.

MOLAR CONDUCTIVITY (Λ) is defined as the quantity of a solution which contains one gramme-molecule of the solute when measured with electrodes one centimetre apart.

For monovalent salts $\lambda = \Lambda$

For divalent salts $2\lambda = \Lambda$

DEGREE OF DISSOCIATION : In an electrolyte the degree of dissociation is obtained as $\alpha = n/N$, where n is the number of molecules dissociated and N is the total number of molecules present in the solution.

Variation of conductivity : It is found that the conductivity of a solution diminishes on dilution. It is as would be expected, as on being diluted a given volume of solution contains a diminished member of ions and since conductivity is due to ions present, conductivity should fall with decrcase of concentration. So if fresh ions are not produced by dilution conductivity should be inversely proportional to the dilution of the solution. Again, since equivalent conductivity measures the conducting power of a fixed amount of the solute it should have a constant value and should not show any decrease with dilution. It is actually found to be approximately constant for some salts but most of the solutions of inorganic salts in water show increased equivalent conductivity with dilution. The behaviour of equivalent conductivity with dilution thus divides the solution in two distinct classes.

WEAK AND STRONG ELECTROLYTES : If only a fraction of total number of molecules of the solute are dissociated and that fraction increases with dilution *i.e.* if fresh ions are produced by dilution the equivalent conductivity increases on diluting and tends towards a superior limit for infinite dilution. There are some electrolytes which behave in this way. These are known as ***weak electrolytes.*** Acetic acid, sodium carbonate solution are electrolytes of the type. These electrolytes obey the ***Ostwald's dilution law***, concerning α, the degree of dissociation and c, the concentration, expressed in symbols in the form, $\frac{\alpha^2 c}{1-c} = \text{constant.}$

If λ_c and λ_∞ are respectively the equivalent conductivities at concentration c and at infinite dilution then,

$$\alpha = \frac{n}{N} = \frac{\lambda_c}{\lambda_\infty}$$

Thus Ostwald's law may be written as

$$\frac{\lambda_c^2 . c}{(\lambda_\infty - \lambda_c)} = \text{constant.}$$

In weak electrolytes λ increases from a low value as the dilution is increased. But yet there is another kind of electrolytes in which α remains practically constant at a high value showing no change with dilution and maintains its value even for a high value of d, though at low values of d, λ is slightly lower. These are known as *strong electrolytes.*

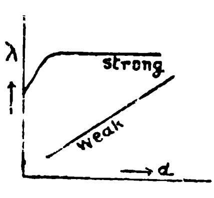


Fig. 4·3

DEBYE-HUCKEL THEORY : The anomalous behaviour as shown by weak and strong electrolytes has been explained by Debye-Huckel theory of complete dissociation. According to this theory strong electrolytes are completely dissociated at all concentrations. In weak electrolytes α varies with concentration and for high concentration $\alpha \rightarrow 0$. In strong

electrolytes α remains practically unchanged with variation of concentration and $\alpha \rightarrow 1$ for all concentrations. Since strong electrolytes are completely dissociated there is no change in equivalent conductivity with change of concentration. The deviation from ideal behaviour, that is slight increase with dilution at low concentrations is due to the decrease in mobilities of ions arising out of retarding forces. These are caused by (i) inter-atomic forces arising out of electrostatic attraction between opposite charges, (ii) the retardation caused by surrounding polar molecules of the solvent and (iii) the viscosity of the liquid. At higher dilutions all these effects become negligible.

IV-4. MIGRATION OF IONS

Ionic Mobility : Ions remaining in free state in a solution experience force due to applied electric field and as such acquire velocity. The maximum velocity depends upon the intensity of the electric field and the charge of the ion. But it is also controlled by the nature of solvent and the effective size of the ion considering that it remains surrounded by a group of neutral molecules Since this varies for different ions, even in an equal electric field the velocity of ions of different substances will not be the same.

Ionic mobility is defined as the velocity of the ion under a potential gradient of one volt per centimetre. It is a constant for a particular kind of ion at a given temperature and concentration. If the velocity is v centimetres per second and the potential gradient is E volts per centimetre then the ionic mobility is obtained as $\mu = v/E$.

Ionic velocity and change in concentration: It is found that there is a variation in concentration of the solution at the anode and that at the cathode as well when a current flows for sometime. This is accounted for by the difference in velocity of two kinds of ions.

Let u and v be respectively the velocities of the positive and negative ions. The current and therefore the amount of

deposit in a given time will be proportional to $u+v$. This will be evident by considering the facts that (*i*) the current is measured as the transference of total charge, positive and negative charges moving in opposite directions producing a a cumulative effect, (*ii*) the deposit at any electrode consists not only of the ions which have migrated there from the other end but also of those set free there by the transport of oppositely charged ions.

Let us consider that the flow of current for such a time that $(u+v)$ gramme-molecules of the solute have been withdrawn from the solution. For this $(u+v)$ gramme-ions have been deposited at the cathode and of this u gramme-ions have been gained by migration from the other end leaving a net loss of $(u+v)-u=v$ gramme-ions of positive ions of the solution around the cathode. Also v gramme-ions of negative ions have been lost by transport to the other end. Hence there is a net loss of v gramme-molecules of solute in solution in the region of the cathode. Similarly it may be argued that on the anode side round about it there has been a loss of u gramme-molecules of solute. Hence

$$\frac{\text{Loss of concentration at the cathode}}{\text{Loss of concentration at the anode}}=\frac{v}{u}$$

It is to be noted that since $(u+v)$ gramme-molecules of solute has been removed from the solution the net loss of concentration of solution as a whole is $(u+v)$ gramme-molecules. This deduction is based on the assumption that there is a middle compartment in which the concentration remains unchanged. But prolonged flow of current and mechanical disturbance inside the solution destroys this ideal state.

The reductions in concentrations as stated, do not occur if the electrodes are soluble. This happens when the electrodes have the same metallic radical as that of the solute, such as in the use of silver electrodes in the electrolysis of silver nitrate or of copper electrodes in copper sulphate solution. In such a case, the total amount of solute in the solution

remains unchanged, there being gain in concentration near the anode accompanied by equal loss around the cathode. Let us consider the electrolysis of $CuSO_4$ solution with copper electrodes at a stage when $(u+v)$ gramme-ions of copper has been deposited on the cathode. This is the loss of copper of the solution around the cathode. This region has gained u gramme-ions of copper by migration from the other end. Hence net loss of copper is $(u+v)-u=v$ gramme-ions. Since the loss of SO_4 ions by transport to the other end is also v, so the net loss of solute in the cathode region is v gramme-molecules. At the anode copper goes into solution and net gain of copper is $(u+v)$ gramme-ions. Of this u gramme-ions has been removed by transport. Hence the net gain of copper ions is v gramme-ions. Also an equal quantity of SO_4 ions is gained by migration. Hence net gain of $CuSO_4$ in the solution around the cathode is v gramme-molecules.

Transport Numbers : The quantity of charge carried by positive ions per second across unit area of the solution is proportional to u and the charge carried by negative ions in the same time across the same area is proportional to v. Hence the current flowing through the solution is proportional to $(u+v)$. The fractions of the total current carried by cations and anions are called their respective transport numbers and are respectively equal to

$$\frac{u}{u+v} \text{ and } \frac{v}{u+v}.$$

It is also evident that, $\dfrac{u}{u+v}$

$$=\frac{\text{Mass of metallic ion removed from the solution near cathode}}{\text{Total mass of metallic ion deposited on the cathode.}}$$

Ratio of velocities of anions and cations *i.e.* v/u may be obtained by measuring the change in concentration at the two electrodes. This can be done by special experimental arrangement. Two beakers filled with electrolyte are electrically connected by a small siphon containing the electrolyte. This siphon arrangement prevents the diffusion of the solute

tending to equalise any difference in concentration. If a current is passed for sometime there will be changes in concentration of the solutes in the two beakers in which the respective electrodes are dipped. This change may be determined by chemical methods. The ratio of the losses of concentrations determines $\frac{v}{u}$.

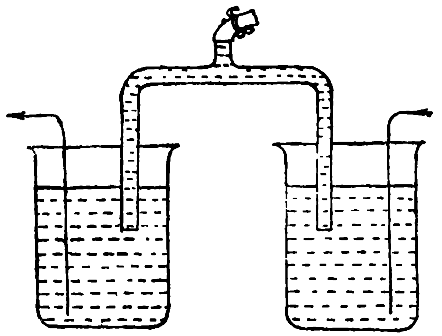


Fig. 4·4

Determination of absolute ionic velocities : The current through an electrolyte is carried by ions. Consider the case in which the ions are monovalent, so that one gramme-equivalent of each kind of ion conveys a charge of 96470 coulombs. Let there be n gramme-equivalents of dissociated molecules per cubic centimetre of the solution. Then the concentration (c) of both kinds of ions are $1000n$ gramme-equivalents per litre. Denoting the velocities of positive and negative ions by u and v respectively, the charges of two kinds passing unit cross-section of the solution in one second are respectively $n.u.96470$ and $n.v.96470$ coulombs. So the effective current density is $n(u+v)$ 96470 coulombs.

The magnitude of current may also be calculated by applying Ohm's law. Let ρ be the specific resistance of the electrolyte and V the potential drop across the electrolyte between the electrodes. If l be the length of the electrolyte and A its cross-section then R, the resistance of the electrolyte contained in this volume, is given by

$$R=\rho.\frac{l}{A}$$

Denoting the specific conductivity by K, since $\rho=\frac{1}{K}$

$$R=\frac{l}{KA}$$

So the current $i=\frac{V}{R}=\frac{VAK}{l}$

Current-density $I=\frac{i}{A}=\frac{VK}{l}=K.e$, e being the fall of potential per unit length.

So $n(u+v)\ 96470=K.\ e.$

$$\text{or} \quad u+v=\frac{K}{n}\cdot\frac{e}{96470}$$

Since $1000n=c$, $u+v=\frac{K}{c}\cdot\frac{e}{96\cdot47}=\frac{K}{c}\times0\cdot1036c$

Since $\frac{K}{c}=\lambda$, $(u+v)=0\cdot1036\lambda.e$

If $e=1$ volt/cm, $(u+v)$ becomes the sum of ionic mobilities of positive and negative ions. λ occuring in the equation for $(u+v)$ may be determined by Kohlrausch's method (*See IV*-9).

Thus we find that v/u and $(u+v)$ may be determined experimentally and so u and v may be obtained.

IV-5. REVERSIBLE CELLS

Characteristics of a reversible cell: A voltaic cell is regarded as a reversible one if it satisfies the following conditions.

(i) When the cell is connected to an external source of *emf* exactly equal to that of the cell, chemical action within the cell ceases and the cell fails to send a current.

(ii) If the external *emf* is slightly less than the *emf* of the cell, a small current flows from the cell. On the other hand, if the external *emf* exceeds that of the cell by the same small amount, the same current flows through the cell in the opposite direction and all reactions occuring through the cell when it produces current, are reversed.

In a cell, such as a Daniell cell, in which there is no polarisation, the ions liberated do not alter the nature of electrodes. Such a cell behaves as a reversible cell. But even in such a cell, if the current flows for a considerable time,

heat is produced in the cell and diffusion takes place inside the solution. Both of these are irreversible processes. So a reversible cell should be regarded as such when very small charge flows from it.

A Daniell cell may be constructed by placing a zinc electrode in aqueous solution of $ZnSO_4$ and a copper electrode in $CuSO_4$ solution in water. The two solutions are kept separated by a porus pot. The cell may be represented chemically as

$$Zn^- \mid \underrightarrow{ZnSO_4\ aq.} \vdots \underrightarrow{CuSO_4\ aq.} \mid Cu^+$$

The positive current flows from zinc to copper through the solutions. During the passage of current zinc dissolves in solution and appears there are as Zn^{++} *ions*, while at the other end Cu^{++} ions are deposited on the copper electrode from the solution. If an opposing *emf* slightly greater than that of the cell is applied, current flows in the opposite direction through the cell causing deposit of zinc on the zinc-electrode and dissolving of copper-electrode. These two reactions are reversible and may be indicated symbolically as shown below

$$Zn + Cu^{++} \xrightarrow[\text{external emf in action}]{\text{cell in action}} Zn^{++} + Cu$$

If the two *emfs* are equal the reactions cease.

Application of Thermodynamics : A reversible cell may be regarded as reversible heat engine and its *emf* may be calculated by applying the laws of thermodynamics.

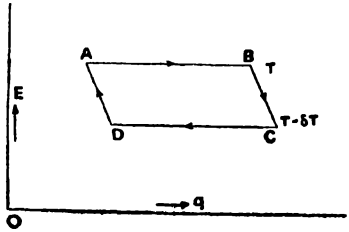


Fig. 4·5

Consider a reversible cell whose *emf* E is kept in a balance by an equal and opposing *emf* from an external source. The cell is in an enclosure kept at a temperature $T°K$. By slightly reducing the

balancing *emf* let the cell be caused to supply a charge q for a small interval of time δt. The straight line A-B in the E-q-indicator diagram (Fig. 4·5) represents the passage of charge q isothermally (E as such remaining constant) at $T°K$. If the energy generated by chemical action in the cell is not sufficient to supply the electrical energy necessary to drive the charge, heat is absorbed. Since the process is isothermal the temperature remains constant and external source supplies the heat necessary. Let h joules be the heat supplied by the surroundings.

Now let the cell be thermally isolated and again by slightly reducing the balancing *emf* the cell be made to pass a small charge dq. This process being adiabatic, the cell absorbs heat from within and falls in temperature. Let the temperature be $T-\delta T$ and let the fall in temperature cause a fall in *emf* to $E-\frac{dE}{dt}\,\delta T$. In the diagram BC represents this operation.

Next let the balancing *emf* be increased, so that it is slightly greater than the *emf* of the cell. Let a charge q pass isothermally through the cell in the reverse direction. CD represents this in the indicator diagram. An amount of heat is liberated in the process and it is absorbed by the surroundings.

Finally, pass a charge dq through the cell adiabatically so that the temperature and the *emf* of the cell are restored to their initial values (T and E respectively). DA in the diagram represents this.

The four operations as described constitute a reversible cycle. If W be the useful work obtained and h the amount of heat drawn at the higher temperature and δT is the temperature difference then we may write by applying the second law of thermodynamics

$$\frac{W}{h}=\frac{\delta T}{T}.$$

The work done by the cell represented by AB in the diagram is Eq joules and the work done on the cell represented

by $CD=\left(E-\frac{dE}{dt}\,\delta T\right)q$. So the net useful work obtained in the cyclic operation is given by

$$W=q.\frac{dE}{dt}\,\delta T$$

(the work along BC and DE are equal and opposite)

Hence $$\frac{q\frac{dE}{dt}\,\delta T}{h}=\frac{\delta T}{T}$$

or $$h=qT.\frac{dE}{dT}\text{ joules}$$

h is the heat drawn from the source when a charge q coulombs passes. This energy is supplied in addition to that obtained by chemical action. If H joules be the heat liberated when one coulomb is generated Hq joules are evolved during the passage of q coulombs. So equating the work done by the *emf* and heat energy supplied for the same we may write

$$Eq=Hq+h=Hq+q.T.\frac{dE}{dT}$$

or $$E=H+T.\frac{dE}{dT}.$$

This equation is known as **Gibbs-Helmholtz equation.**

If $E>H$, that is if the heat generated by chemical action is short of the energy (as supplied by the *emf*) of the cell, the cell while working therefore absorbs its own heat and falls in temperature there being a corresponding fall of *emf* with temperature *i.e.* $\frac{dE}{dT}$ is positive. If on the other hand it is observed that heat is generated when the cell is in action and current flows, then $E<H$ and $\frac{dE}{dT}$ is negative. So the *emf* decreases with rise of temperature. For a Daniell cell $\frac{dE}{dT}$ is zero and hence $E=H$. Therefore *emf* of the Daniell cell may be calculated directly from a knowledge of H, as shown in the following lines.

E. M. F. OF A DANIELL CELL : The electro-chemical equivalents of copper and zinc are respectively 0·00033 and 0·00034 gm/coulomb. When one gramme of zinc dissolves in acid 1630 calories of heat are liberated and when one gramme of copper is deposited 881 calories are absorbed. Hence the balance of heat generated when one coulomb of charge flows, expressed in joules, is obtained as

$$(1630 \times 0{\cdot}00034 \times 4{\cdot}2) - (881 \times 0{\cdot}00033 \times 4{\cdot}2) = 1{\cdot}1 \text{ joules}$$

Energy required to pass a coulomb of charge through a potential difference of E volts is $E \times 1$ joules, hence equating the two energies

$E \times 1 = 1{\cdot}1$, or $E = 1{\cdot}1$ volts, which is the *emf* of the Daniell cell.

Standard cell : There are two other reversible cells used as standards of *emf*. In these cells constancy of *emf* is obtained and there is very little variation with temperature.

WESTON-CADMIUM CELL : The anode is mercury with a layer of mercurous sulphate on it. The cathode is a cadmium amalgam. The electrolyte is the saturated solution of cadmium sulphate. Cadmium sulphate crystals are kept near the cathode to maintain the saturation. The chemicals are contained at the bottom of two limbs of a glass tube shaped like *H*. Platinum electrodes are sealed through the bottom of each limb. In one branch mercury stands over the platinum wire, this forms the positive terminal. There is a paste of mercurous sulphate and cadmium sulphate covering the mercury surface. In the other limb mercury and cadmium amalgam covers the platinum wire forming the negative electrode and over it there are some cadmium sulphate crystals. The rest of the tube contains a saturated solution of cadmium

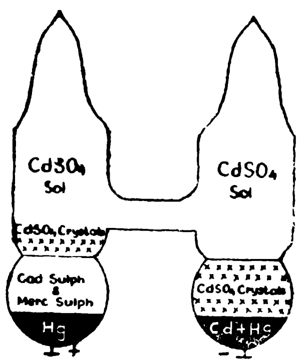


Fig. 4·6

sulphate. The upper ends of the limbs are closed by fusing in a flame. The arrangement as stated may be represented as

$$\overset{+}{\text{Hg.}} \;\Big|\; \underset{\text{Solid}}{\mathbf{Hg_2SO_4}} \;\vdots\; \underset{\text{Sol.}}{\mathbf{CdSO_4}} \;\Big|\; \overset{-}{\textbf{Cd. amalgam}}$$

$$\longleftarrow\!\!-\!\!-\!\!-$$

When the cell is in action cadmiun ion from the amalgam comes into solution and reacting with Hg_2SO_4 liberates $2Hg^+$ which is deposited on the anode

$$Cd^+ + Hg_2SO_4 \longrightarrow CdSO_4 + 2Hg^+$$

The *emf* of the Weston-Cadmium cell is 1·01830 volts at 20°C. Any variation of the *emf* with temperature ·is indicated by the expression

$$E = E_{20} - 40{\cdot}6 + 10^{-6}(t-20) - 9{\cdot}5 \times 10^{-7}(t-20)^2 + 10^{-8}(t-20)^3 \text{ volts.}$$

LATIMER-CLARK CELL : In this cell zinc amalgam is used as the negative electrode, the electrolyte is the saturated zinc sulphate solution. Mercury forms the positive electrode. The components may be expressed in symbols, as shown

$$\overset{+}{\text{Hg}} \mid Hg_2SO_4 \;\vdots\; ZnSO_4 \mid \overset{-}{\text{Zn-amalgam}}$$

The reaction in cell consists of solution of zinc in the electrolyte and deposit of mercury from Hg_2SO_4 on the positive electrode. The *emf E* at any temperature *t* is given by

$$E = 1{\cdot}438\,[1 - 1{\cdot}19 \times 10^{-5}\,(t-15) - 7 \times 10^{-6}(t-15)^2] \text{ volts.}$$

The *emf* has a higher temperature coefficient than that of the cadmium cell.

IV-6. CONCENTRATION CELLS

Principle : Helmholtz first pointed out that instead of utilising the chemical action as a source of energy, the diffusion occuring between two solutions of the same substance at different concentrations may be used in construction of cells. The electrodes are made of same metal. The arrangement of such a cell may be represented as follows :

Copper Cell :	$\overset{+}{Cu}$	**$CuSO_4$ sol. conctd.**	**$CuSO_4$ sol. dilute**	$\overset{-}{Cu}$
Silver Cell :	$\overset{+}{Ag}$	**$AgNO_3$ sol. conctd.**	**$AgNO_3$ sol. dilute**	$\overset{-}{Ag}$

In such an arrangement the metal in contact with dilute solution goes into solution and that inside the concentrated solution gets a deposit. Thus the direction of conventional current within the cell is from the dilute to the concentrated solution. The metal in contact with the dilute solution forms the negative terminal of the cell when the cell is connected to an external circuit.

Change of concentration of solution : Let $(u+v)$ gramme-atoms or gramme-equivalents of the metal be dissolved in the dilute solution and an equal amount deposited from the concentrated solution, u and v being the ionic velocities of positive and negative ions respectively. We may consider the loss or gain of concentration of the solutions round the two electrodes for the silver cell as shown below. Let $(u+v)$ gramme-atoms (which is same as the gramme-equivalent for silver) be deposited on the cathode.

Inside the concentration solution

Loss of *Ag*. by deposition	...	$(u+v)$ gm.-equiv.
Gain of *Ag*. by migration	...	u gm.-equiv.
Net loss of $Ag.=(u+v)-u$	...	v gm.-equiv.
Also loss of NO_3 by transport	...	v gm.-equiv.
Hence loss of $AgNO_3$	...	v gm.-mols.

Inside the dilute solution

Gain of *Ag* by solution	...	$(u+v)$ gm.-equiv.
Loss of *Ag* by transport	...	u gm.-equiv.
Net gain of $Ag=(u+v)-u$	...	v gm.-equiv.
Also gain of NO_3 by migration	...	v gm.-equiv.
Hence gain of $AgNO_3$	...	v gm.-mols.

Thus when $(u+v)$ gramme-equivalents of *Ag* are deposited on the positive plate of the cell, there is a net transference v gramme-molecules of $AgNO_3$ from the concentrated to the

dilute solution. Hence for a deposit of one gramme-equivalent of silver the transference will be of $v/(u+v)$ gramme-molecules of $AgNO_3$.

Source of energy : The energy of current in a concentration cell is obtained from diluting of the solution. Work is performed as a solute in solution expands from a volume v_1 to v_2. This is due to osmotic pressure exerted by a substance in solution. This pressure has the same magnitude as that exerted by an equal number of molecules existing as a gas in a space equal in volume to that of the solution. The gas law $pv=RT$ is applicable to solutions, considering p as the osmotic pressure and v as the reciprocal of c, the concentration in gramme-molecules per unit volume. Consider a solution which by the action of an osmotic pressure p expands in volume by δv in a reversible way. The work done is $p.\delta v$. The total work for a change from a volume v_1 to a volume v_2 is obtained as

$$W=\int_{v_1}^{v_2} p.dv.=\int_{v_1}^{v_2} \frac{RT}{v}dv=RT\left[\log_e \frac{v_2}{v_1}\right]$$

Since $v_1=1/c_1$ and $v_2=1/c_2$

$$W=RT\log_e\left(\frac{c_1}{c_2}\right)$$

Remembering that when the solution is completely dissociated the osmotic pressure is double of that for no dissociation we shall write $W=2RT\log_e\left(\frac{c_1}{c_2}\right)$

E.M.F. OF THE CELL : Work performed in transference of charge q by an *emf* E is Eq. This energy is obtained from the process of dilution. If q is the charge carried by n gramme-equivalents

$$Eq=2nRT\log_e\left(\frac{c_1}{c_2}\right)$$

For a gramme-equivalent of deposit $q=9647$ *e.m.* units and $n=v/u+v$, the transport ratio of the negative ion. Hence E is obtained.

Temperature coefficient : In a concentration cell there is no chemical change and all the processes are reversible. Applying Gibbs-Helmholtz equation $E=H+T\frac{dE}{dT}$ to a concentration cell, H should be put equal to zero, because there is no chemical reaction causing liberation or absorption of heat, so

$$E = T\frac{dE}{dT}$$

$$\text{or} \quad \frac{dE}{E} = \frac{dT}{T}$$

$$\text{or} \quad \log_e\left(\frac{E}{T}\right) = \text{constant}, \quad \text{so} \quad E \propto T.$$

Thus the *emf of the concentration cell is proportional to the absolute temperature.*

Amalgam concentration cells : Two electrodes of amalgam having different concentrations of a metal dipped in an electrolyte of a solution of same salt of the metal form another type of concentration cell. The electromotive force of such a cell causes a transference of the metal from the amalgam of greater to that of lower concentration since the osmotic pressure in an amalgam is proportional to concentration, the work done by a solute in expanding is given by

$$\int_{p_1}^{p_2} p.dv = \int_{v_1}^{v_2} \frac{RT}{v}.dv = RT \log_e\left(\frac{v_2}{v_1}\right)$$

If each molecule dissociates into n other on going into solution the work is

$$W=nRT \log_e\left(\frac{v_2}{v_1}\right)=nRT \log_e\left(\frac{p_1}{p_2}\right)$$

p_1 and p_2 are the osmotic pressures of the metal in the amalgam and in the solution respectively.

If E_1 is the *emf* and qr is the quantity of charge which passes from the concentrated amalgam to the solution for

transference of one gramme-atom, the work is $E_1 qr$, where $q=9647$ (*e.m.u*) and r is the valency, so

$$E_1 = \frac{nRT}{9647r} \log_e \left(\frac{p_1}{p_2} \right)$$

Similarly at the other electrode

$$E_2 = \frac{nRT}{9647r} \log_e \left(\frac{p_3}{p_2} \right)$$

The resulting *emf* $E = E_1 - E_2 = \dfrac{nRT}{9647r} \log_e \left(\dfrac{p_1}{p_3} \right)$

It may be mentioned that concentration cells have little practical use.

IV-7. ELECTRODE POTENTIAL

Solution pressure : When a metal is placed in a solution of its salt, as copper in $CuSO_4$ solution, the osmotic pressure of the solution drives the metallic ions upon the metals causing a deposit. On the contrary atoms on the metal are urged by a solution pressure which tends to drive them as ions into solution. When these two opposing forces are equal an equilibrium is set up. For every solution there is a particular osmotic pressure acting on the metallic ions for which neither the ions are deposited on the solid, nor the ions from the solution goes into solution. At this stage, the solution pressure P is equal to the osmotic pressure p. If P be either greater or less than p, two different state of things occur.

(i) If $p > P$, the result is that more ions are deposited than the number going into solution. As a result the solid becomes gradually more and more positively charged with respect to the solution and thereby a potential difference is created. Soon a stage is reached when the positive potential of the solid becomes so high that it prevents further incoming of the positively charged ions.

Such a condition is obtained when a copper plate is dipped in $CuSO_4$ solution. A definite potential difference is established between the solution and the copper plate which remains at

a higher potential and is taken as positive with respect to the solution.

(ii) If $p<P$, the metal ultimately remains negatively charged with respect to the solution. Such a condition is obtained when a zinc plate is dropped into $ZnSO_4$ solution.

Such a potential at the solid-liquid interface is called *electrode potential.*

Nernst's equation for electrode potential : Let us consider that a metal with a valency r is inserted in an electrolyte containing its own ions. Let P be the solution pressure and p the osmotic pressure and $P>p$. Let N gramme-atoms pass from the metal and be dissolved in the liquid. Considering that 9650 e. m. units of charge are associated with one gramme-equivalent, the charge carried by N gramme atoms is $N \times r \times 9650$. Let π be the electrode potential (e.m. units). A positive *emf* is directed from metal to the solution, since $P>p$. The work involved in the transference is obtained as

$$W=\pi Nr \times 9650 \text{ ergs}$$

Consider the gas law that may be applied here. If one gramme-atom of a gas occupies a volume v under a pressure P, the work done by the gas when it expands isothermally by δV is $P.\delta v$. Since Pv=constant, $P.dv= -v.dP$. Hence the energy released by N gramme-atom of gas when its volume changes so as to cause a fall in pressure from P to p may be obtained as

$$W=N\int_1^2 P.dv= -N\int_P^p v.dP=NRT\int_p^P \frac{dP}{P}=NRT \log_e \left(\frac{P}{p}\right)$$

Applying this gas law to the change of pressure of the ions when that comes from the metal into the solution and equating the energy released to the work done for the transference of charge, we get

$$\pi Nr \times 9650=NRT \log_e \left(\frac{P}{p}\right)$$

$$\text{or} \quad \pi = \frac{RT}{9650r} \log_e \left(\frac{P}{p}\right) \text{ e. m. units}$$

This equation is known as **Nernst equation** for the electrode potential. If $P>p$, π is regarded as positive, it being directed from the metal to the solution as in the case of zinc. On the other hand, if $P<p$, π is considered negative acting from the solution towards the metal, as in the case of copper. The *emf* remains active so long as its action exceeds the counter force due to the potential difference established by transference of ions.

Putting $R=8{\cdot}314\times10^7$, π at 25°C calculated in volts is obtained as

$$E=\frac{8{\cdot}314\times10^7\times298}{9650\times r\times10^8}\log_e\left(\frac{P}{p}\right)$$

$$\text{or}\quad E=\frac{0{\cdot}058}{r}\log_e\left(\frac{P}{p}\right)\text{ volts.}$$

The reactions at the electrodes are reversible and these are therefore called *reversible electrodes.* There are two types of these electrodes. In the first type reversibility concerns the transference of cations and in the second type it refers to to anions. Calomel electrode described hereafter forms an electrode of the second type.

In a voltaic cell two electrodes having oppositely directed *emfs* are placed together in such a way as to produce a cumulative effect. In a Daniell cell the electrodes are both of the first type and in a Cadmium cell one electrode is of the first type and the other of the second type.

A positive value of π obtained from the Nernst equation indicates a potential directed from metal to the solution. The *emf* of a voltaic cell is the algebraic difference of the two electrode potentials. In a Deniell cell, for zinc in $ZnSO_4$ solution, $P>p$ and for copper in $CuSO_4$ solution $P<p$, hence the two *emfs* may be written as

$$E_1=\frac{0{\cdot}058}{r}\log_e\left(\frac{P_1}{p_1}\right)\quad\text{and}\quad E_2=\frac{0{\cdot}058}{r}\log_e\left(\frac{P_2}{p_2}\right)$$

The *emf* of the cell is $E=E_1-E_2=\dfrac{0{\cdot}058}{2}\left[\log_e\left(\dfrac{P_1}{p_1}\right)+\log_e\left(\dfrac{p_2}{P_2}\right)\right]$

The *emf* is directed from electrode-1 to electrode-2 inside

the cell. The *emfs* in open circuit remain in a balanced condition. But if the two electrodes are connected externally, the balance is disturbed and both the *emfs* become active, as a result zinc goes into solution and copper is deposited on the metal. So far as the external circuit is concerned, copper is the positive terminal but within the cell the *emf* is directed from zinc to copper.

Normal electrode : The electrode potential between a metal and an electrolyte is to be measured with the help of a second electrode, whose electrode potential is known. Such an arrangement is necessary because of the fact that any attempt to measure the electrode potential by introducing a second metal inside the electrolyte defeats its own propose and the measurement gives the algebraic sum of the two electrode potentials. An electrode having no contact potential with the solution would have served the purpose. Electrode formed by mercury drops meets this demand. But it is not convenient for practical purpose, so a *normal electrode* whose electrode potential is known is used. Calomel electrode serves as an electrode for this purpose. It consists of mercury contained in vessel over which a layer of insoluble mercurous chloride is kept and upon it a normal solution of potassium chloride is poured in which fills the vessel. There is a side tube filled with this solution which projects outwards. This can be placed in any cell of which the electrode potential is to be measured. A platimum wire is used to make contact with electrode. The electromotive force of the combination is obtained in the ordinary way ; the unknown *emf* is calculated by deducting the *emf* of the calomel electrode from the observed *emf.*

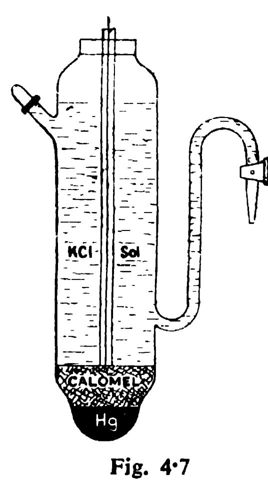


Fig. 4·7

The *emf* of the calomel electrode is 0·56 volt, mercury being positive.

IV-8. SECONDARY CELLS

Accumulators : The disadvantage of using voltaic cells as current sources are that use of these involves continuous loss of materials and they are unsuitable for a large steady current for prolonged use. These cells have been superseded by a class of cells called *accumulators* which store or accumulate electric energy obtained from a powerful source. These are called *secondary cells* in-as-much as they require primary charging before they can serve as current giving cells. Accumulators are of two types—Acid cell or Lead accumulator and Alkali or Ni-Fe cell.

Lead Accumulators : When dilute sulphuric acid (H_2SO_4) is electrolysed between two lead plates, the anode becomes coated with lead-peroxide (PbO_2) while the cathode remains unaltered. On breaking the circuit and connecting the plates with a conducting wire, a current is found to flow through the wire. This current arises out of polarisation. This principle is utilised in lead-acid accumulators.

Two formed plates containing $PbSO_4$ are taken and dipped into a solution of H_2SO_4. When a current is passed through the electrolyte, hydrogen ion (H^+) liberated at the cathode reduces lead sulphate to lead and the anode receiving anions is converted into lead peroxide (PbO_2) by the action of SO_4 radicals. The reactions are as shown below.

Reactions during charging

At the positive plate

$$\mathbf{PbSO_4+SO_4^{--}+2H_2O=PbO_2+H_2SO_4+2e^-}$$

At the negative plate

$$\mathbf{PbSO_4+2H^+=Pb+H_2SO_4+2e^+}$$

The plates are now said to be charged. A potential difference has been established between the plates (now dissimilar) and if they are joined externally a current flows until the plates are restored to their original condition. The reactions are as shown below.

Reactions during discharge

At the positive plate

$$\mathbf{PbO_2+2H^++H_2SO_4=PbSO_4+2H_2O+2e^+}$$

At the negative plate

$$Pb+SO_4^{--} = PbSO_4+2e^-$$

It may be observed that during charging the cell gains H_2SO_4 and during discharge the electrolyte gains water (H_2O). The specific gravity of acid indicates the condition of the cell.

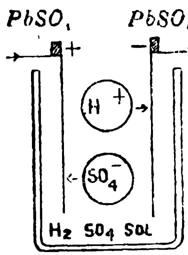


Fig. 4·8a

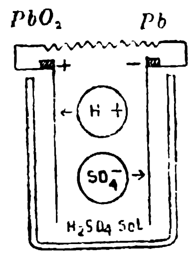


Fig. 4·8b

The *emf* of a fully charged cell is about 2·2 volts. The internal resistance is small. So the cell should be always used with some external resistanee and should never be short-circuited.

Alkaline Accumulators : The active materials in this type of storage cells (also called Edison cells) are nickel-oxide in the positive and iron-oxide in the negative plate. The electrolyte is 21 percent solution of potassium hydroxide (KOH) with addition of a small quantity of lithium hydrate, which increases the capacity of the cell. The oxides all remain as hydrated. The exact formula of nickel-oxide is not yet established. The action of the cell can be explained by considering the hydroxides.

A charged cell has $Ni(OH)_4$ in the positive plate and Fe in the negative plate. During discharge when the cell supplies current (OH) ions of KOH travel to the negative electrode and ion is oxidised in the hydrated form. The K-ions travel to the positive plate and reduce $Ni(OH)_4$ to $Ni(OH)_2$. During charging a reverse action takes place. The reactions can be represented by the reversible equation

$$Ni(OH)_4+KOH+Fe \underset{\text{Charge}}{\overset{\text{Discharge}}{\rightleftarrows}} Ni(OH)_2+KOH+Fe(OH)_2$$

The electrolyte does not take part in any chemical change but simply acts as a carrier for transfer of (OH) ions from one plate to another.

The *emf* of the cell is 1·3 volts. It has several advantages over the lead accumulators. It takes less time for charging. It is capable of maintaining a large current and is insensitive to mechanical vibrations. It has a larger life and is not easily damaged by overcharging. It has, of course, higher initial cost, a lower voltage and slightly higher resistance in companion with lead cells.

Ampere-hour capacity : The total quantity of charge a storage cell can deliver after full charging is measured as its ampere-hour capacity. The product of the current in amperes and the time in hours for which it can supply current is a measure of the capacity of the cell.

IV-9. APPLIANCES AND MEASUREMENTS

Silver Coulometer : The mass of any substance liberated at an electrode by electrolysis is obtained from the expression $m=Zq=Zit$. Hence if Z, *i.e.* the electro-chemical equivalent is known, the measurement of amount of deposit (m) in a known time (t) determines the current (i) causing the liberation. The apparatus for this purpose is known as *Voltameter* or *Coulometer*.

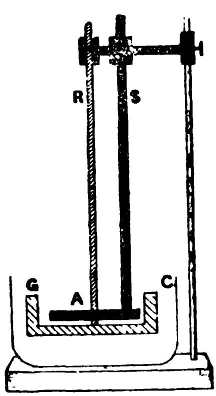


Fig 4·9

The essential parts of a silver coulometer are a silver disc (A) placed in a glass disc (G) which again is held in position by a glass rod (R). A silver rod (S) attached to silver disc serves as the current lead. The silver disc forms the anode and the cathode is a platinum bowl (C) which contains ten percent solution of silver nitrate. When a current is passed through the solution, silver is deposited in the bowl. The gain in its mass after electrolysis is obtained for the desired determination of current from the formula $i=m/Zt$.

DETERMINATION OF E.C.E. OF SILVER : By obtaining the mass (m) of deposit in time t by a known current(i) Z can be obtained from the expressions $m=Zit$. The value of the current should be very accurately determined for this purpose. This may be done by a potentiometer. A standard low resistance (r) is inserted in the coulometer circuit and the potential drop at its ends is determined. If e be the potential drop, $e=ir$, hence i is obtained. The mass of deposit in the cathode is obtained by weighing the platinum bowl before and after the experiment. The current should be passed through an hour or so.

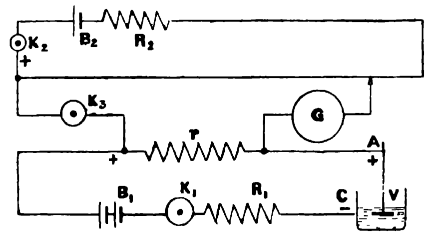


Fig. 4·10

Determination of Equivalent Conductivity : The resistance of a column of electrolyte if measured in the ordinary way is not obtained accurately due to polarisation effect. The liberated gases which are frequently the products of electrolysis increase the resistance between the electrodes and also set up a counter *emf*. Futher, loss of concentration by the passage of current through appreciable time also leads to wrong result due to variation of conductivity with concentration. A satisfactory method for avoiding these effects consist in using of a high frequency source of *emf*. The reversal of current many times a second completely removes polarisation and prevents loss of concentration.

KOHLRAUSCH'S BRIDGE : Kohlrausch used alternating current in a specially designed wheatstone bridge arrangement consisting of a metre bridge and a non-inductive variable resistance. A small induction coil or a valve oscillator, either supplying an intermittent or a high frequency (1000 *c.p.s.*) current is used as the source of *emf*. The electrolyte is taken in a glass tube with rubber corks through which the electrodes pass. The separation between the electrodes effecting a variation of length of column of electrolyte may be adjusted to

different values. A pair of earphones is used in place of galvanometer to respond to the alternating current. Perfect balance cannot be obtained due to stray capacitances. As the jockey slides along the bridge wire, the sound in the phone varies in intensity and the minimum sound is taken to indicate null condition of the bridge. If an electrolytic resistance be S, then with R as the third arm resistance, for a balance at a length l of the bridge wire measured from the R-end, the relation between the resistances are

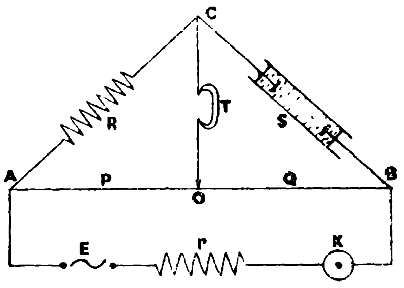


Fig. 4·11

$$\frac{R}{S}=\frac{l\sigma}{(100-l)\sigma},$$ σ being the resistance per unit length of the bridge wire.

Hence $\dfrac{R(100-l)}{l}=S=\rho\dfrac{L}{A}$, · So $\rho=\dfrac{AS}{L}$,

ρ is the specific resistance of the solution, L is the length and A is the cross-section of the liquid column.

If K is the specific conductivity and λ the equivalent conductivity,

—then $\lambda=\dfrac{K}{c}=\dfrac{AS}{Lc}=\dfrac{A}{Lc}\cdot\dfrac{R(100l)}{l}$

c is the concentration. Determining λ for different values of c, the variation of λ with c may be obtained. This is shown approximately by the λ-c graph (Fig. 4·12).

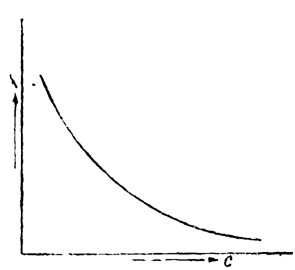


Fig. 4·12

NUMERICAL EXAMPLES

1. A current of 2·68 amperes is passed through an aqueous solution of $CuSO_4$ *for one hour. Calculate the gramme-equivalents of copper deposited.*

[Given $F=96480$ coulombs]

Solution : Charge passed $q = 2{\cdot}68 \times 60 \times 60 = 9648$ coulombs

$$\text{Hence deposit} = \frac{q}{F} = \frac{9648}{96480} = 0{\cdot}1 \text{ gm-equiv.}$$

2. *Calculate the number of molecules per cubic centimeter of hydrogen at N.T.P. from the following data :*

Density of hydrogen at N.T.P. $= 9 \times 10^{-5}$ *gm/c.c.*

E.C.E. of hydrogen $= 104 \times 10^{-7}$ *gm/coulomb*

Charge of monovalent ion $= 1{\cdot}6 \times 10^{-19}$ *coulomb*

Solution: Mass of 1 c.c. of hydrogen $= 9 \times 10^{-5}$ gm.

Charge required to liberate 1 c.c. of hydrogen from a compound calculated as mass/*E.C.E.* is

$$\frac{9 \times 10^{-5}}{104 \times 10^{-7}} = \frac{9}{1{\cdot}04} \text{ coulombs}$$

$$\text{Number of atoms in 1 c.c. of hydrogen} = \frac{9}{1{\cdot}04 \times 1{\cdot}6 \times 10^{-19}}$$

$$\text{Hence number of molecules} = \frac{9 \times 10^{19}}{2 \times 1{\cdot}04 \times 1{\cdot}6} = 2{\cdot}704 \times 10^{19}$$

3. *A reversible cell has an emf of 1·0032 volts at 10°C and heat developed in the chemical reactions in the cell is 0·26 calories/coulomb. Colculate the change in emf of the cell if its temperature is raised to 15°C.* [*Given J = 4·2 joules/calorie*]

$$\textit{Solution}: \quad E = H + T\frac{dE}{dT}$$

$$1{\cdot}0032 = 4{\cdot}2 \times 0{\cdot}26 + 283\frac{dE}{dT}$$

$$\text{Hence } \frac{dE}{dT} = -\frac{0{\cdot}0928}{283}$$

$$\text{Therefore } E_{15} = 4{\cdot}2 \times 0{\cdot}26 + 290\left(-\frac{0{\cdot}0928}{283}\right) = 1{\cdot}0016 \text{ volts.}$$

4. *If 1 gm. of hydrogen when forming water liberates* 35×10^3 *colories of heat, calculate the back emf produced in a water voltameter.*

[*Given E.C.E. of Hydrogen* $= 1{\cdot}04 \times 10^{-5}$ *gm/coulomb, Equivalent weight of hydrogen* $= 1{\cdot}008$ *and 1 Faraday* $= 96500$ *coulombs*]

Solution : 1 gm.-equiv. of hydrogen requires $35 \times 10^3 \times 1{\cdot}008 \times 4{\cdot}2$ joules of energy for electrolysis. If e is the back *emf* and F is the charge carried by 1 gm-equiv. of hydrogen, the energy used is eF. Equating these two

$$eF = 35 \times 10^3 \times 1{\cdot}008 \times 4{\cdot}2$$

$$\text{or} \quad e = \frac{35 \times 10^3 \times 1{\cdot}008 \times 4{\cdot}2}{96500} = 1{\cdot}53 \text{ volts.}$$

EXERCISES ON CHAPTER IV

4·1. Discuss the mechanism of electrolytic conduction in the light of Arhenius' theory. What are the evidences in support of this theory ?

4·2. State and explain Faraday's Laws of electrolysis.

Explain the terms : Electro-chemical equivalent, Faraday, International Ampere.

What is Back *emf* in electrolysis ? What is its effect ?

4·3. Give an account of the theory of electrolytic conduction and explain how Faraday's laws of electrolysis follow from this theory.

Discuss how electrolytic conduction differs from conduction through a metal.

4·4. Explain the mechanism of conduction in an electrolyte. Discuss how polarisation is minimised in the measurement of conductivity in an electrolyte.

4·5. Discuss the variation of electrolytic conductivity with concentration.

What are weak and strong electrolytes ? State how Debye-Huckel theory explains the difference in behaviour of two kinds of electrolyte.

Explain the terms equivalent conductivity, degree of dissociation, concentration and dilution.

4·6. Define ionic mobility. Describe how ionic velocities may be experimentally determined.

4·7. What are the requisities of reversible cell ?

Deduce Gibbs-Helmholtz equation concerning the *emf* of a reversible cell and explain its significance.

4·8. What is a concentration cell ? Obtain an expression for its *emf* and show that it is proportional to absolute temperature.

4·9. What are the characteristics of a standard cell ?

Describe any type of standard cell mentioning the conditions that have to be satisfied.

4·10. What is electrode potential ? Explain how it is formed and discuss its importance in the construction of a voltaic cell.

Deduce Nernst's equation for electrode potential and hence obtain an expression for the *emf* of a Daniell cell.

4·11. Explain the terms Solution pressure and Osmotic pressure. Deduce how these two create a potential difference between a metal and a solution of its salt.

What is a normal electrode ? Describe any form of it. What is its special use ?

4·12. What is a secondary cell ? Describe any form of it. What are its advantages ?

4·13. Describe a method of determining the equivalent conductivity of a solution.

4·14. Describe a silver coulometer and the determination of electro-chemical equivalent of silver with its help. Show how it measures current.

4·15. Calculate the equivalent conductivity of an electrolyte if the mobilities of the ions concerned in a completely dissociated state are $6{\cdot}6\times10^{-4}$ and $5{\cdot}8\times10^{-4}$ centimetres per second per volt per centimetre. [*Ans* : 0·197/ohm-cm.]

4·16. The electro-chemical equivalent of hydrogen is $1{\cdot}04\times10^{-5}$ gm/coulomb. Calculate the charge on a monovalent ion assuming Avogadro number to be 6×10^{23}.

[*Ans* : $1{\cdot}603\times10^{-19}$ coulomb]

4·17. If u and v be the ionic mobilitics in electrolysis prove that

$$u+v=0{\cdot}1036\lambda,$$

where λ is the equivalent conductivity.

CHAPTER V

ELECTRO-MAGNETIC INDUCTION

V-1. MAGNETIC LINKAGES

Magnetic Flux: Consider any closed area in a magnetic field. Some lines of induction will pass through this area according to the position of the area with respect to the direction of induction. If **B** is the area with respect to the direction of induction at every point in an element $\delta\mathbf{S}$, then $\mathbf{B}.\delta\mathbf{S}$ is considered to be the flux through $\delta\mathbf{S}$. If θ be the angle made by the normal to the area (δS) with the direction of induction (B), then the flux through the area δS is $B.\delta S \cos\theta$. The flux through a finite boundary is expressed as

$$N = \int \mathbf{B}.\delta\mathbf{S}.$$

Maxwell turns: When a coil consists of n turns each of equal area and wound closely together, the effective flux through the coil is given by

$$n\int \mathbf{B}.\delta\mathbf{S} = nN$$

The expression determines the total magnetic linkage with the coil and is expressed in *maxwell-turns*. The unit of flux in the *c.g.s.* system is *maxwell* and the practical unit is *weber*.

Note: It may be helpful to remember different units concerning magnetic field and flux and their relationship. Magnetic field or magnetising force is expressed in *Oersteds* (dynes per pole). Magnetic flux is measured in *maxwell*. *Gauss* in the unit of induction or flux-density. So, 1 maxwell per square centimetre is 1 gauss. *Weber* is the practical unit of flux. 10^8 maxwells = 1 weber. 10^8 gauss relates to a flux-density of 1 weber per square centimetre.

Change of flux through a coil causes induced *emf* whose magnitude is determined by the rate of change of flux. Weber is that flux which when changed in one second induces an

emf of 1 volt in each turn of a coil. Hence weber is measured as volt-second per turn.

V-2. LAWS OF ELECTRO-MAGNETIC INDUCTION

E. M. F. **generated by a changing flux** : While investigating whether a current element produces in a neighbouring circuit any effect in a manner similar to electrostatic and magnetic inductions, Faraday observed that so long as the current in a circuit remains steady, there is no effect in any neighbouring circuit in the form of production of any current. But any change in the strength of current causes a transient current to flow in any neighbouring circuit which is called a secondary circuit with respect to the primary circuit, which itself is carrying a current. Further, if a conductor producing a magnetic field or even a magnet approaches to or recedes from a closed circuit a current is found to flow through the circuit so long as the motion continues.

Faraday's Laws : It may be observed that in all cases of production of transient current in a closed circuit (having no source of *emf* included in it), there is a change of flux linked with the circuit. This led Faraday to enunciate his laws of electro-magnetic induction stating that (*i*) *when the magnetic flux linked with a closed circuit changes, an electromotive force acts round the circuit,* (*ii*) *the magnitude of the emf thus induced is determined by the rate of change of flux and its direction is determined by the nature of change, increase or decrease.*

Neumann's Law : Faraday's experimental deduction regarding magnitude of induced *emf* was expressed by Neumann by a law leading to a mathematical equation.

The induced emf in a circuit when the magnetic flux through it changes is equal to the rate of variation of magnetic flux.

Lenz's Law : The direction of the induced current is obtained by a principle enunciated by Lenz :

The direction of the induced emf is such that it tends to oppose the cause of variation of magnetic flux, in other words it tends to keep the magnetic flux unchanged.

These two laws can be expressed in a mathematical form. If e is the magnitude of induced *emf* and δN is the change of flux through the circuit in time δt, then

$$e = -\frac{\delta N}{\delta t}\text{, and in the limit } e = -\frac{dN}{dt}$$

FARADAY'S LAWS FROM PRINCIPLE OF CONSERVATION OF ENERGY : The magnitude of the induced *emf* as given by the equation $e = -\frac{dN}{dt}$ may be deduced from the principle of conservation of energy. Two particular cases may be considered such as (i) a magnetic pole is approaching a closed circuit and (ii) a closed circuit is moving in a constant magnetic field.

Case I : Let a closed circuit AB carrying current i and having a radius r be placed in a field due to a magnetic pole of strength $+m$ at P (Fig. 5·1). The direction of the current as observed from P is clockwise *i.e.* the face of the equivalent magnetic shell representing s-pole is turned towards P.

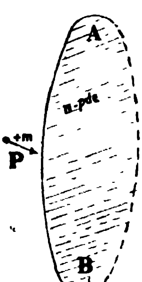


Fig. 5·1

Consider the magnetic potential at P due to the equivalent magnetic shell. If it be Ω, then $\Omega = i\omega$, where ω is the solid angle subtended at P by the boundary of the circuit. Let the action of the current cause the pole to move through a distance δx towards AB in time δt, so there is a change of solid angle by an amount $\delta\omega$. Consequent change of potential is $\delta\Omega = i.\delta\omega$. The work done on the pole for the shift is $m.\delta\Omega = mi.\delta\omega$, $\delta\omega$ is positive since m approaches towards AB. The energy supplied by the source having an *emf* E is $Ei.\delta t$. This is spent in two parts, if r be the resistance of AB, $i^2r\delta t$ for heat production and $mi.\delta\omega$ for the work done on the pole, hence

$$Ei.\delta t = i^2 r\delta t + mi.\delta\omega$$

$$\text{or } i = \frac{1}{r}\left[E - m\frac{\delta\omega}{\delta t}\right]$$

$m\frac{\delta\omega}{\delta t}$ is a quantity which as appears in the equation for current acts in opposition to E. This is the magnitude of the induced *emf* caused by the relative motion between the magnetic pole and the current carrying circuit. Since the total flux around the pole is $4\pi m$, the flux linked with the circuit is obtained as $N=4\pi m.\frac{\omega}{4\pi}=m\omega$. Hence

$Lt.\ \frac{m\delta\omega}{\delta t}=\frac{d(m\omega)}{dt}=\frac{dN}{dt}$, so the magnitude of induced *emf* is $e=-\frac{dN}{dt}$.

Case II-A : Let an elementary closed circuit of area δS carry a current i. The magnetic moment of the equivalent shell is $\mu i.\delta\mathbf{S}$, μ being the permeability of the medium. Magnetic potential energy of the shell in a field H is $\mu i\mathbf{H}\delta\mathbf{S}$. Let the action of the field cause a motion of the coil, thereby effecting a loss of potential energy. Since the coil moves from weaker to the stronger part of the field, let the field in which the coil is shifted to, be $\mathbf{H}+\delta\mathbf{H}$. Hence the loss of potential energy is given by

$\mu i\,(\mathbf{H}+\delta\mathbf{H}).\ \delta\mathbf{S}-\mu i\mathbf{H}.\delta\mathbf{S}=i.\mu\delta\mathbf{H}.\delta\mathbf{S}.=i\delta\mathbf{B}.\delta\mathbf{S}$ where $\delta\mathbf{B}$ is change in induction through the coil. Consider the elementary shell to be part of a closed circuit of finite boundary. Since a magnetic shell of finite boundary has the same magnetic effect as due to all the elementary shells in which the entire shell may be divided, hence the total work done by the circuit of finite size is obtained as $i\int\int d\mathbf{B}.d\mathbf{S}=i.\delta N$, where δN is the change of flux through the coil.

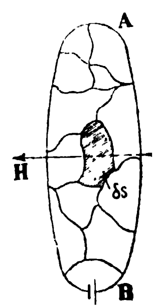


Fig. 5·2

Equating the energy supply from the source of *emf E* in time δt in a circuit of resistance r to the sum of energy consumed for heating and that required for the shift we may write,

$$Ei.\delta t = i^2 r\delta t + i.\delta N$$

$$\text{or } E = ir + \frac{\delta N}{\delta t} = ir + \frac{dN}{dt} \text{ in the limit}$$

$$\text{Hence } i = \frac{1}{r}\left[E - \frac{dN}{dt}\right]$$

So the induced *emf* opposing the applied *emf* is $\frac{dN}{dt}$.

Case II-B : Consider the case in which instead of entire circuit being free to move, a part of it is so. Taking a simplified case in which the length δl of a conductor *PQ* forms a part of a closed circuit carrying a current i and having a resistance r. This conductor *PQ* when acted upon by a force at right angles to its length slides upon parallel conducting rails *Cc*, *Dd* which form parts of circuit.

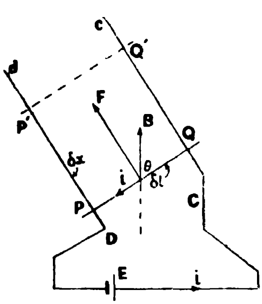


Fig. 5·3

Let the circuit be in a field of magnetic induction **B**, acting at right angles to the plane of the paper and inclined at an angle θ with *PQ*. The force on *PQ* for its being placed in the magnetic field is

$$\mathbf{F} = i\mathbf{B} \times \delta\mathbf{l} = iB.\delta l.\sin\theta = iH\delta l.\sin\theta, \text{ in air.}$$

If it causes a displacement δx, the work done is

$$W = \mathbf{F}.\delta\mathbf{x} = i(\mathbf{B} \times \delta\mathbf{l}).\delta\mathbf{x} = i(\delta\mathbf{x} \times \delta\mathbf{l}).\mathbf{B}$$

$$\text{or } W = i.H.\delta l \sin\theta.\ \delta x \text{ in air.}$$

Since $H.\delta l.\delta x.\sin\theta = \delta N$, the flux embraced by PQ during the shift, hence $W = i.\delta N$.

Equating the energy supplied by the source in time δt, with the work done in production of heat and mechanical shift, we may write

$$Ei.\delta t = i^2 r\delta t + i.\delta N$$

$$\text{or } i = \frac{1}{r}\left[E - \frac{\delta N}{\delta t}\right] = \frac{1}{r}\left(E - \frac{dN}{dt}\right) \text{ in the limit.}$$

Hence the induced *emf* is $e = -\frac{dN}{dt}$.

V-3. INDUCED E.M.F. IN SOLID CONDUCTORS

Foucault or Eddy Current : Besides being induced in closed coils, currents are also produced by induction in solid conductors which are subject to varying flux. These are called *Eddy currents.* The energy concerning such a current is dissipated in the form of heat inside the conductors. The temperature of a metal is raised considerably when rotated rapidly between the poles of a strong magnet. So flow of eddy current dissipates considerable energy. This presents a serious difficulty in designing of armatures of dynamos and motors, which are caused to rotate in a magnetic field. To avoid production of eddy currents the armature cores, instead of being made of solid iron, are in laminated form, consisting of discs insulated from one another. In such an arrangement eddy currents which tend to flow at right angles to the magnetic field are diminished in intensity as they are prevented from having a low resistance path (see § VIII-2, Fig. 8·4).

Arago's Disc : The production of eddy current can be demonstrated in a simple experiment. If a copper disc placed

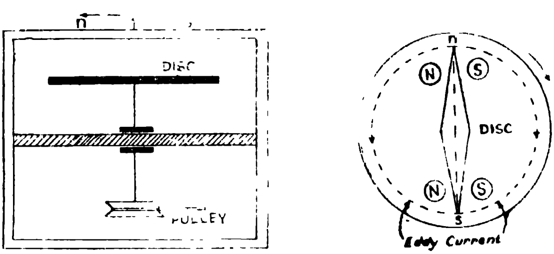


Fig. 5·4

just below a pivotted magnetic needle is rotated round a vertical axis passing through the pivot of the needle the magnet is found to rotate in the same direction as the disc.

The reason for such a behaviour of the magnetic needle is that induced currents are set up in the copper disc moving in the magnetic field. The circuit may be considered to be completed along a diameter and the circumference of the disc, By Lenz's law the direction of the induced current is such

that it tends to oppose the motion producing it. The relative motion between the disc and the magnet is the cause of the current and the magnet rotating in the same direction as the disc tends to minimise this. It may also be explained in a different way. The magnet which rests in the magnetic meridian experiences a couple due to the magnetic field caused by the induced currents in the copper plate and hence a motion. For a continuous rotation of the copper disc the couple due to the induced magnetic field must be greater than due to earth's field. This may be obtained by increasing the speed of rotation.

For similar induced current an oscillating magnet suspended above a stationary metal plate soon comes to rest. Based on this principle, a moving coil galvanometer is made dead beat by winding its coil on a metal frame or by short-circuiting the coil of the galvanometer. The effect of induced current due to motion of the coil in magnetic field brings the coil to rest.

Faraday's Disc : Faraday devised an arrangement in which the eddy current can be transferred to an external circuit causing deflection in a galvanometter.

A copper disc is rotated between the pole pieces of a horse-shoe magnet and a circuit is completed through a galvanometer by means of two metal springs, one of which presses on the axle and the other on the outer edge of the disc. When the disc rotates, any radius of the disc at any instant connected through the springs with the outer circuit sends the induced current through the galvanometer.

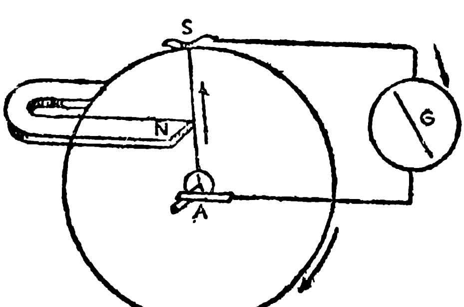


Fig. 5·5

If the radius of length r rotates n-times per second in a field

H, the area swept out by the radius per second is $n\pi r^2$ and the flux cut by it is $n\pi r^2 H$. This is the magnitude of the induced *emf*.

V-4. SELF-INDUCTION

Self-inductance : When a current is being established in a coil, there is setting up of magnetic field inside. The corresponding flux is enclosed by the coil itself and according to the principle of electro-magnetic induction there should be an induced current superposed on the main current. By Lenz's law this induced *emf* is in opposition to the *emf* applied. This phenomenon of generation of *emf* is known as the ***self-induction*** of the circuit.

By Lenz's law the induced *emf* will tend to weaken a growing current and strengthen one that is decaying. As a result the current due to a steady *emf* applied to a coil does not assume the value demanded by Ohm's law immediately on closing the circuit. Again, there is also a tendency of the current to continue even after the withdrawal of the *emf* and as a consequence a spark is created across the gap formed by break of the circuit.

Coefficient of Self-induction : The flux (N) linked with a circuit is proportional to the current (i) flowing and so

$$N \propto i \quad \text{or} \quad N = Li$$

L is a constant depending upon the dimensions of the coil and is known as ***coefficient of self-induction*** or ***self-inductance*** of the coil. If in the above expression i is put equal to 1, then N becomes equal to L, thus ***self-inductance of a circuit can be numerically equated to the flux linked with unit current flowing in it***, provided there is no ferro-magnetic substance inside it.

Again, if e is the induced *emf* due to a change of flux through the circuit, then

$$e = -\frac{dN}{dt} = -\frac{d}{dt}(Li) = -L\frac{di}{dt}$$

If $\frac{di}{dt} = 1$, $e = L$. Thus ***self-inductance may be considered to***

be numerically equal to the emf induced in it due to unit rate of change of current in the circuit.

Further, when a current i is growing in a circuit, the induced *emf* e acts in opposite direction thus delaying the establishment of a steady current. The growing current has to do a work amounting to $e.i$ per second against the opposing *emf* at any instant when the current is i. Total work done when a steady current i_0 is established in time t is obtained as

$$W=\int_0^t ei.dt=-L\int_0^t i\frac{di}{dt}dt=-L\int_0^{i_0} i.di=-\tfrac{1}{2}Li_0{}^2$$

This is the energy stored up in the magnetic field created by the current, due to self-inductance of the coil. When the circuit is broken the energy is returned to the coil and it appears as a spark.

If $i_0=1$, W becomes equal to $\frac{1}{2}L$, or $L=2W$. Hence ***self-inductance may be obtained numerically as twice the work done in establishing the flux linked with unit current in the circuit.***

The three values of L obtained from different considerations are invariable so long as the permeability of the medium inside the coil does not change. But if the permeability varies with change in the field, the three definitions are not identical and the values are not constant. It may be evident from the evaluations of L in such a case as shown below. Let A be the area enclosed by a coil of n turns carrying a current i. The field H may be expressed as ki, k being a constant. Let μ be the permeability.

$$\text{Flux } N=AnH\mu=Anki\mu$$

$$\text{So } N=Li=Anki\mu, \text{ hence } L=Ank\mu.$$

Consider that μ varies with H and H varies with i,

$$\text{Hence } \frac{\partial N}{\partial i}=Ank\left(\mu+i\frac{\partial\mu}{\partial i}\right)$$

$$\text{So } e=-\frac{dN}{dt}=-\frac{\partial N}{\partial i}.\frac{\partial i}{\partial t}=-Ank\mu\left(1+\frac{i}{\mu}.\frac{\partial\mu}{\partial i}\right)\frac{\partial i}{\partial t}$$

But $e = -\frac{Ldi}{dt}$, Hence $L = Ank\mu\left(1 + \frac{i}{\mu} \cdot \frac{\partial\mu}{\partial i}\right)$

UNIT OF SELF-INDUCTANCE in practical units is **Henry** measured as the induced *emf* in volts for a change of current in the circuit at the rate of one ampere per second. Since 1 volt$=10^8$ *e.m.* units of *emf* and 1 ampere$=10^{-1}$ *e.m.* unit of current, 1 Henry$=10^9$ *e.m.* units of self-inductance.

Calculation of Self-inductance : In particular cases as in symmetrical circuits, self-inductance may be calculated from the knowledge of magnetic field due to a current flowing in the circuit.

SELF-INDUCTANCE OF CIRCULAR COIL : The field inside a circular coil of radius a and n turns is given by $H = 2\pi ni/a$. If N be the total flux through the coil, then

$$N = \pi a^2 nH = \pi a^2 n . \frac{2\pi ni}{a} = 2\pi^2 an^2 i$$

$$L\frac{di}{dt} = \frac{dN}{dt} = 2\pi^2 an^2 \frac{di}{dt}, \text{ hence } L = 2\pi^2 n^2 a.$$

If μ is the permeability of the medium inside, then $L = 2\pi^2 n^2 a\mu$. This value is applicable if the field inside is uniform and equal to that at the centre.

SELF-INDUCTANCE OF A SOLENOID : In the case of a *long solenoid* of total length l, radius a ($l >> a$) and of cross section $A = \pi a^2$, having n turns per unit length and carrying a current i, the field inside is given by $H = 4\pi ni$ and the magnetic induction $B = 4\pi ni\mu$. Hence the flux linked with each turn is given by $4\pi ni\mu A$. The total flux (N) through all the turns is obtained as $N = 4\pi\mu lAn^2 i$.

Induced *emf* for a change in current is

$$e = -\frac{dN}{dt} = -4\pi\mu lAn^2\frac{di}{dt}$$

$$\text{Since } e = L \text{ when } \frac{di}{dt} = 1, \text{ So } L = 4\pi\mu lAn^2$$

In air, $L = 4\pi n^2 Al = 4\pi^2 n^2 a^2 l$.

It has been shown that for a *short solenoid* of length l and

radius a, at a distance x from the centre at a point on the axis the field inside is obtained as (§ II-2)

$$H=2\pi ni\left[\frac{x}{\sqrt{x^2+a^2}}+\frac{l-x}{\sqrt{(l-x)^2+a^2}}\right]$$

Flux linked with an element δx at x in a medium of permeability μ is

$$\delta N=\mu\pi a^2 n.\delta x.2\pi ni\left[\frac{x}{\sqrt{x^2+a^2}}+\frac{l-x}{\sqrt{(l-x)^2+a^2}}\right]$$

Hence the flux linked with the whole solenoid is

$$N=2\pi^2n^2a^2\mu i\left[\int_0^l\frac{x.dx}{\sqrt{x^2+a^2}}+\int_0^l\frac{(l-x)dx}{\sqrt{(l-x)^2+a^2}}\right]$$

For evaluating $\int_0^l\frac{x.dx}{\sqrt{x^2+a^2}}$, put $z=x^2+a^2$, so $dz=2x.dx$

Hence the integrand$=\int\frac{dz}{2\sqrt{z}}=\sqrt{z}$

Hence $\int_0^l\frac{xdx}{\sqrt{x^2+a^2}}=\left[\sqrt{x^2+a^2}\right]_0^l=\sqrt{l^2+a^2}-a$

Similarly the other part will also be $\sqrt{l^2+a^2}-a$

Hence $N=4\pi^2n^2a^2\mu i\,[\sqrt{l^2+a^2}-a]$

So $L=4\pi^2n^2a^2\mu\,[\sqrt{l^2+a^2}-a]$

Note : *If $l<10a$, the above formula is only approximate ; for such short solenoids L is calculated from Nagaoka's formula $L=4\pi^2n^2alk$ (e.m.u), where k is a function of a/l, and its value is obtained from worked out results.*

SELF-INDUCTANCE OF PARALLEL WIRES : Let A and B be two parallel wires at distance d (centre to centre) apart. Suppose the currents in the both the wires are i but flowing in mutually opposite directions. The supply wires from a source to a load is an example of such an arrangement. Let a be the radius of each of the wires ($a<<d$).

Considering that the field at a distance r from a straight

wire carrying a current i is $H=\frac{2i}{r}$, the flux through an area of length l and width δx (Fig. 5·6) at a distance x from one wire is given by

$$N=2il\int_a^{d-a}\left(\frac{1}{x}+\frac{1}{d-x}\right)dx = 2il\left[\log_e\left(\frac{d-a}{a}\right)-\log_e\left(\frac{a}{d-a}\right)\right]$$

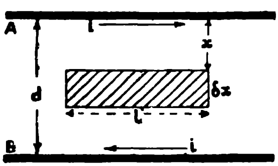


Fig. 5·6

or $N=Li=4l\,\log_e\left(\frac{d-a}{a}\right)i$

Self-inductance L is given by

$$L = 4l\,\log_e\left(\frac{d-a}{a}\right)$$

If the wires are very close so that $(d-a)\rightarrow a$, $L=0$. This fact gives the principle of winding of non-inductive resistances. The insulated wire is first doubled and then the resistor coil is prepared with the two-fold wire.

SELF-INDUCTANCE OF CO-AXIAL CYLINDERS : Let us calculate the self-inductance of two cylinders of radius a and b ($a < b$) placed co-axially one inside the other. When both the cylinders carry current we may consider that the magnetic field in the gap between the two is due to the current (i) in the inner cylinder, since there is no field inside a hollow conductor. The field at distance r from the axis of the inner cylinder is $H=2i/r$ and the flux through unit length of the shaded area (Fig. 5·7) bounded by the two cylinders is given by

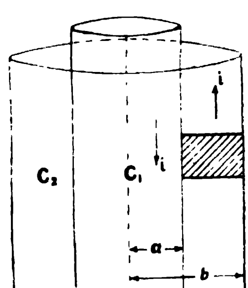


Fig. 5·7

$$N=\int_a^b\frac{2i}{r}\,dr = 2i\Big[\log_e r\Big]_a^b=2i\,\log_e\left(\frac{b}{a}\right)$$

Hence the self-inductance per unit length of the cylinder is $L = 2\log_e\left(\frac{b}{a}\right)$ and in a medium of permeability μ

$$L = 2\mu\,\log_e(b/a).$$

Such co-axial conductors are found in the type of cable in which the inner wire is the 'lead' and the outer conductor is the 'return' of the circuit. The field outside the cable is zero, since the two cylinders carry current i in mutually opposite directions. These are used for transmitting radio-frequency alternating current as the cable becomes a non-radiating conductor.

V-5. MUTUAL INDUCTION

Mutual Induction : A variation of current in one circuit is accompanied by an induced *emf* in a neighbouring circuit. This is the phenomenon of *mutual induction.*

The flux N linked with a circuit when a current i flows through a neighbouring circuit is proportional to the current, that is

$$N \propto i, \quad \text{or} \quad N = Mi,$$

M is a constant, called the *co-efficient of mutual induction* or *mutual inductance* of the two coils. In the foregoing equation ($N=Mi$) if i is put equal to 1, then N becomes equal to M. So *mutual Inductance is numerically equal to the flux passing through one coil (secondary) due to unit current in other (primary).* It is a constant for any pair of coils in a particular configuration irrespective of the current flowing provided that no ferro-magnetic substance is within or near the coils. In presence of such a substance M does not remain proportional to the current owing to variation of permeability of ferro-magnetic material and consequent change in flux.

Induced *emf* in the secondary circuit is given by $e = -\dfrac{dN}{dt}$.

Since $N = Mi$, $e = -M\dfrac{di}{dt}$. If $\dfrac{di}{dt} = 1$, e is numerically equal to M, so *mutual inductance may also be measured as the numerical value of the induced emf for unit rate* of current change in the other. Unit of mutual inductance is henry, same as that of self-inductance.

Mutual potential energy of two coils : Let us consider two circuits A_1 and A_2 in a medium of permeability μ carrying

currents i_1 and i_2 respectively. The strength of the equivalent magnetic shells are respectively μi_1 and μi_2.

Potential energy of a magnetic shell is given by

$W=$(strength of the shell) $\times$ (lines of force through it).

Hence considering the two coils the potential energy of the one due to the other is obtained as shown.

If the effective flux through A_1 due to a current in A_2 is N_2, then the potential energy of A_1 due to A_2 expressed as W_{12} is given by

$$W_{12}=(\mu i_1)\times\left(\frac{N_2}{\mu}\right)=N_2 i_1$$

Similarly if the effective flux through A_2 due to a current in A_1 is N_1, the potential energy of A_2 due to A_1 is

$$W_{21}=\mu i_2\times\left(\frac{N_1}{\mu}\right)=N_1 i_2$$

If again, the mutual inductance of A_1 due to A_2 is denoted by M_{12}, then $N_2=M_{12}i_2$ and so with similar notation $N_1=M_{21}i_1$.

So the mutual potential energy of each of the coils are respectively $M_{12}i_2i_1$ and $M_{21}i_1i_2$. Since these are equal we have $M_{12}=M_{21}$. This means that flux of magnetic induction through A_1 due to unit current in A_2 is same as the flux of magnetic induction through A_2 due to unit current in A_1. Hence if mutual inductance of two coils be represented by M, then their mutual potential energy when they carry currents i_1 and i_2 respectively, is given by Mi_1i_2.

Positive and Negative mutual inductances : If two coils of self-inductance L_1 and L_2 respectively are connected in series so that the effective flux through the two coils are in the same sense

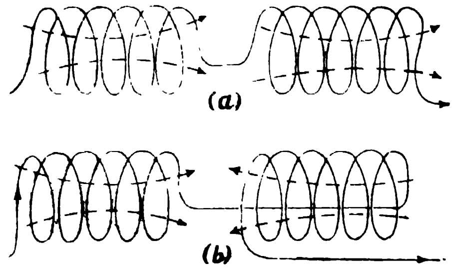


Fig. 5·8

(Fig. 5·8-a) mutual inductance of two is positive. But if the

flux in the two are in opposite sense (Fig. 5·8-b) the mutual inductance is negative. In the first case if the two coils carry the same current i, the flux linkages are respectively

$$L_1 i + Mi \quad \text{and} \quad L_2 i + Mi.$$

The effective self-indutance of the two coils (joined in series) is $L = L_1 + L_2 + 2M$.

If the mutual inductance is negative, the flux linkages are $L_1 i - Mi$ and $L_2 i - Mi$ and the effective self-inductance in such a case is $L' = L_1 + L_2 - 2M$. Hence $L - L' = 4M$.

Coefficient of Coupling : Let two coils having n_1 and n_2 as total number of turns respectively be so placed that the effective flux of one is completely linked with the other. If N is the flux through each turn of the coil-1, the current in which is i_1, the self-inductance of the coil defined as the linkage of flux for unit current in it is given by

$$L_1 = \frac{n_1 N}{i_1} \quad \text{or} \quad N = \frac{L_1 i_1}{n_1}$$

The mutual inductance M_{21} defined as the flux through coil-2, due to unit current in coil-1 is obtained as

$$M_{21} = \frac{n_2}{i_1} . N = \frac{n_2}{i_1} . \frac{L}{n_1} i_1 = L_1 \frac{n_2}{n_1}$$

(It is obtained with the assumption that flux linkage with each turn of either coil is same.)

Similarly if i_2 is the current in coil-2, N' is the flux through each turn, and L_2 its self-inductance

$$\therefore \quad L_2 = \frac{n_2 N'}{i_2} \quad \text{or} \quad N' = \frac{L_2}{n_2} i_2$$

$$\text{Hence } M_{12} = \frac{n_1}{i_2} N' = L_2 \frac{n_1}{n_2}$$

Writing $M_{21} = M_{12} = M$, $M^2 = L_1 L_2$ or $M = \sqrt{L_1 L_2}$. Of course in practice this condition is never satisfied. The ratio $\frac{M}{\sqrt{L_1 L_2}}$ is known as *Coefficient of coupling*. For complete

theoretical coupling the ratio should be unity. For a loose coupling obtained in practice $\sqrt{L_1 L_2} > M$.

Mutual inductance of two solenoids: If one solenoid is wound upon another, we have what is known as **magnetic coupling.** Let the inner solenoid, called the primary (P), contain n_1 turns *per unit length* and carry a current i amperes. Let the outer coil, called secondary and containing n_s turns *in all*, be wound near the central portion of the primary.

The magnetic field (H) inside the primary is given by $H=4\pi n_1 i$. If the primary is wound on a core of material of permeability μ, the magnetic induction B inside is given by $B=\mu H=4\pi\mu n_1 i$. The flux linked with each turn of the

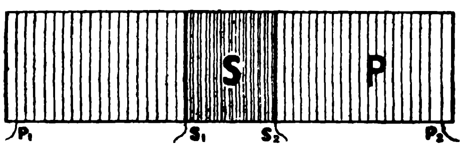


Fig. 5·9

secondary (S) having an area A is equal to $4\pi\mu n_1 iA$. Since there are n_s turns in the secondary, the total flux linked with all the turns is

$$N=4\pi\mu n_1 n_s Ai$$

The induced *emf* in the secondary due to change of flux is given by

$$e=-\frac{dN}{dt}=-4\pi\mu n_1 n_s A\frac{di}{dt}$$

If $\frac{di}{dt}=1$, $e=M$, hence $M=4\pi\mu n_1 n_s A$. In air core $M=4\pi n_1 n_s A$.

If l be the length of the primary, the total number of primary turns is $n_p=n_1 l$. So

$$M=\frac{4\pi n_1 n_s A n_1 l}{n_1 l}=\frac{4\pi n_1{}^2 lA}{n_1 l}.n_s$$

or $\quad M=4\pi n_1{}^2 lA.\frac{n_s}{n_p}$

Self-inductance of thc primary coil is $L=4\pi n_1{}^2lA$, hence $M=\frac{n_s}{n_p}.L.$

If the secondary be a closed circuit the effective inductance of the solenoid is reduced due to opposing flux induced in the secondary.

Standard solenoids are constructed on this principle and their mutual inductance is calculated from the dimensions of the two coils

V-6. INDUCTION COIL

Secondary emf produced by interrupted primary current : Let us consider a circuit in which a battery supplies a current to a primary coil having an arrangement for automatic make-and-break of the circuit (Fig. 5·10a). The primary flux is embraced by a secondary coil Make-and-break of the primary current induces *emf* in the secondary. When the primary current is steady the secondary *emf* is zero Since the primary circuit has an inductance and a resistance the current in it grows and decays exponentially (§ VI-I) as shown in the diagram (Fig. 5·10b). The induced secondary current is somewhat oscillatory in nature flowing alternately in opposite directions. During growth of the primary current the secondary current

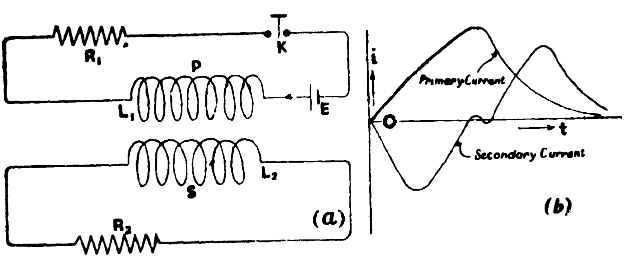


Fig. 5·10

is in the direction opposite to that of the primary but during break of the circuit it is in the same direction with it.

This principle is utilised in a modified arrangement in which the secondary *emf* at the start of the primary current becomes negligible and at break this *emf* is obtained in an amplified form.

Ruhmkorff's Induction Coil : In this apparatus a low *emf* applied at the ends of a primary coil of a few number of turns of thick wire is transformed into an intermittent high potential difference obtainable at the ends of a secondary winding having a large number of turns of fine wire.

The primary (P) is wound on a bundle of soft iron rods (I) forming the core. The primary coil is in series with a make-and-break arrangement consisting of a spring (Sp) in contact with a screw (Sc). The spring carries a soft iron hammer (H). This is attracted by the soft iron core when it is magnetised due to the primary current.

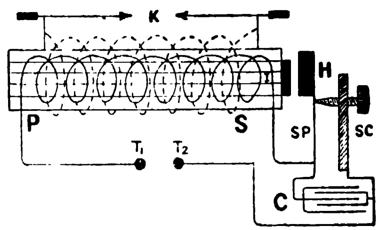


Fig. 5·11

The contact between the screw-head and the spring breaks due to the attraction on the hammer-piece. The primary current is broken and the current ceases to flow. The iron core is demagnetised and the spring falls back again to its normal position in contact with the screw. Hence when a battery is connected, one terminal being joined with the hammer through the primary and the other directly with the screw, an intermittent current passes through the coil. The frequency of make-and-break depends upon the action of the spring.

Over the primary coil there are windings of the secondary coil (S) insulated from the primary. The ends of the coil are connected with two knobs (K) separated by a sparking gap. The rapid make-and-break of the primary circuit produces a changing flux through the secondary and a strong *emf* is induced there. The induced *emf* in the secondary may be obtained as $e = -M\frac{di}{dt}$, where $\frac{di}{dt}$ is the rate of change of current in the primary and M is the mutual inductance.

The primary current is much more effective at 'break' than at 'make' of the circuit in inducing the secondary *emf*. The primary resistance is small and the time constant (L/R) during

growth (see §VI-1) is considerable. When the primary circuit is 'broken' the resistance in the circuit tends to be infinity and the time constant is now negligibly small. Hence $\frac{di}{dt}$ is very high at break, so the *emf* at break is considerably high. But the sudden collapse of the current causes a sparking at the contact point on the screw-head.

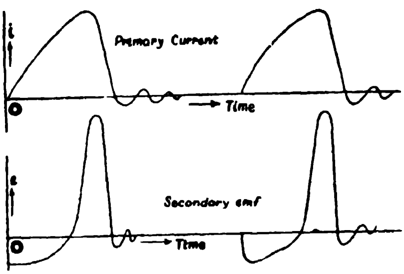


Fig. 5·12

This of course damages the contact surface. Such an arcing is prevented by placing a condenser across (C) the contact point. The *emf* obtained at the ends of the secondary is thus unidirectional though intermittent. The large number of secondary turns, the strong magnetic field within the core and rapid collapse of primary current causes the high *emf* in the secondary.

The effect of the condenser on the secondary may be estimated by comparing the strength of the current in the secondary with and without the condenser.

The *emf* equations of the primary and secondary during break may be written as

$$L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} + i_1 R_1 = 0 \qquad \cdots \quad \cdots \quad \text{(i)}$$

$$L_2 \frac{di_2}{di} + M\frac{di_1}{dt} \qquad = 0 \qquad \cdots \quad \cdots \quad \text{(ii)}$$

L_1, i_1 and R_1 are the primary inductance, current and resistance. L_2, i_2 are the secondary inductance and current. M is the mutual inductance. For simplification the secondary resistance is being neglected in presence of its inductance which is the prominent factor here.

Multiplying (i) by L_2 and (ii) by M and then by subtraction we get,

$$(L_1 L_2 - M^2)\frac{di_1}{dt} + L_2 R_1 i_1 = 0 \qquad \cdots \quad \cdots \quad \text{(iii)}$$

This is an equation of the type $\alpha\frac{dx}{dt}+\beta x=0$, which has the solution of the form $x=x_0e^{-\frac{\beta}{\alpha}t}$.

Hence the solution for i_1 is obtained from (iii) as

$$i_1 = i_0 e^{-\frac{L_2M}{L_1L_2-M^2}.t}$$

or $i_1 = i_0e^{-bt}$, where $b=-L_2M/(L_1L_2-M^2)$
and i_0 is the maximum primary current.

Integrating (ii) we get, $L_2i_2+Mi_1=K$ (constant)

when $i_2=0$, $i_1=i_0$, so $K=Mi_0$

So $L_2i_2+Mi_0e^{-bt} = Mi_0$

or $$i_2= \frac{Mi_0}{L_2}(1-e^{-bt})$$

Maximum value of $i_2= \frac{Mi_0}{L_2}$

When a condenser is put in series with the primary the equations of *emf* for the break of primary circuit are

$$L_1 \frac{di_1}{dt}+M\frac{di_2}{dt}+\frac{Q_1}{C} = 0 \quad \cdots \quad \cdots \quad \text{(iv)}$$

$$L_2 \frac{di_2}{dt}+M\frac{di_1}{dt} = 0 \quad \cdots \quad \cdots \quad \text{(v)}$$

Q_1 is charge on the condenser of capacitance C. Primary resistance has been neglected. Considering $i=\frac{dQ}{dt}$ and $\frac{di}{dt}=\frac{d^2Q}{dt^2}$, the above equations may be rewritten as,

$$L_1 \frac{d^2Q_1}{dt^2}+M\frac{d^2Q_2}{dt^2}+\frac{Q_1}{C} = 0 \quad \cdots \quad \cdots \quad \text{(vi)}$$

$$L_2 \frac{d^2Q_2}{dt^2}+M\frac{d^2Q_1}{dt^2} = 0 \quad \cdots \quad \cdots \quad \text{(vii)}$$

Multiplying (v) by L_2 and (vi) by M, we get

$$L_1L_2 \frac{d^2Q_1}{dt^2}+ML_2\frac{d^2Q_2}{dt^2}+L_2\frac{Q_1}{C} = 0$$

$$ML_2 \frac{d^2Q_2}{dt^2}+M^2\frac{d^2Q_1}{dt^2}=0$$

By subtraction, $(L_1L_2 - M^2)\dfrac{d^2Q_1}{dt^2} + \dfrac{L_2}{C}Q_1 = 0$

This is an equation of the type $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$, which represents a simple harmonic motion. The primary current and the charge are there of oscillatory nature. This is the effect of the condenser.

Integrating (v) we get $L_2 i_2 + M i_1 = K$ (constant) when $i_2 = 0$, let i_1 be equal to i_0, then $K = M i_0$.

Hence we get from the above equation the value of i_2 as

$$i_2 = \frac{M}{L_2}(i_0 - i_1)$$

Maximum value of i_2 is determined by the maximum value of $(i_0 - i_1)$ which is $2i_0$, since i_1 oscillates between $+i_0$ and $-i_0$. Hence maximum value of $i_2 = 2M.\dfrac{i_0}{L_2}$.

So we find that with the condenser in circuit the maximum value of i_2 is doubled. In practice, of course it is somewhat less, because in obtaining this we have neglected the resistances of the winding.

V-7. MEASUREMENT OF INDUCTANCE

Determination of Self-inductance : The inductance to be measured is inserted in one of the arms of a wheatstone bridge and a balance is obtained for the steady current, the battery key being closed before the galvanometer key. The galvanometer used is a ballistic one. On closing the battery with the galvanometer key already closed a throw is obtained, due to the induced *emf* e in the arm AB. Then the current due to instantaneous

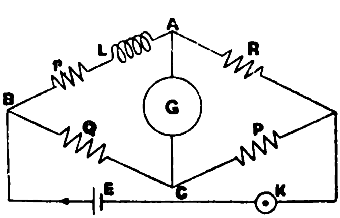


Fig. 5·13

emf $L\dfrac{di}{dt}$ is $KL\dfrac{di}{dt}$ and the total charge flowing through the galvanometer when a current i_0 is established is given by

$$q=\int_0^{i_0} K.L\frac{di}{dt}.dt=KLi_0$$

If the ballistic throw in the galvanometer is θ and λ be the log-decrement then

$$KLi_0=q=\frac{cT}{2\pi AH}\theta\left(1+\frac{\lambda}{2}\right)$$

$$\text{or}\quad L=\frac{c}{Ki_0AH}\cdot\frac{T}{2\pi}\cdot\theta\left(1+\frac{\lambda}{2}\right)\qquad \ldots \qquad \text{(i)}$$

To determine the constant $\frac{Ki_0AH}{c}$, the resistance in AB is changed by a small amount r. This sends a steady current Ki_0r through the galvanometer causing a steady deflection θ_1, given by

$$Ki_0rAH=c\theta_1$$

$$\text{or}\quad \frac{Ki_0AH}{c}=\frac{\theta_1}{r}\qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

From equations (i) and (ii)

$$L=\frac{T}{2\pi}\cdot\frac{r}{\theta_1}\cdot\theta\left(1+\frac{\lambda}{2}\right)$$

Measurement of Mutual Inductance : DEFLECTION METHOD—Let the two coils P and S (Fig. 5·14) having a mutual inductance M be arranged in a circuit as shown in the sketch (Fig. 5·14). ABC is a four-plug key, a small resistor r is joined across B-C so that it falls in series with the primary coil P. With the plug key placed in the gap A, if the primary circuit is closed, a throw will be obtained in the ballistic galvanometer (G) in series with the secondary

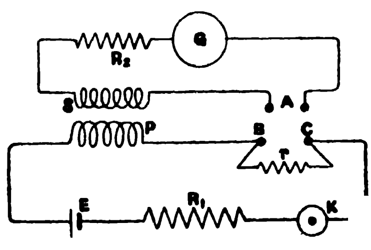


Fig. 5·14

circuit. This throw is due to induced *emf* $M\frac{di}{dt}$. If R is the total

resistance of the secondary circuit the current in it is $\frac{M}{R}.\frac{di}{dt}.$ The charge passing through the galvanometer when a current i_0 is established is given by

$$Q=\frac{M}{R}\int_{0}^{i_0}\frac{di}{dt}.dt=\frac{M}{R}i_0$$

If the flow of charge produces a throw θ in the galvanometer, having a constant $\frac{c}{AH}$ and period of swing T, then

$$\frac{Mi_0}{R}=Q=\frac{T}{2\pi}.\frac{c}{AH}.\theta\left(1+\frac{\lambda}{2}\right),$$ λ being the log-decrement.

The plug in A is taken out and two plugs are placed in the gaps B and C. Now a steady current flows through the galvanometer producing a steady deflection θ_1. This current is of magnitude $\frac{i_0 r}{R}$.

$$\text{So}\quad \frac{i_0 r}{R}=\frac{c}{AH}\theta_1$$

$$\text{or}\quad \frac{c}{AH}=\frac{i_0}{\theta_1}\cdot\frac{r}{R}$$

$$\text{Hence}\quad M=\frac{T}{2\pi}\cdot\frac{r}{\theta_1}\cdot\theta\left(1+\frac{\lambda}{2}\right)$$

NULL METHOD : This method is due to Carey-Foster. The circuit is as shown (Fig. 5·15). The 'make-and-break' of primary current in P causes an induced *emf* in the secondary S and a charge circulates through the ballistic galvanometer. To balance this charge a capacitor C is placed across a resistance r included in the primary circuit. If the windings are proper, the two charges through the galvanometer may be made to flow in mutually opposite directions. The resultant charge may be made equal to zero by adjusting the

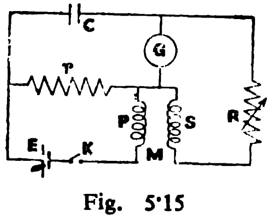


Fig. 5·15

value of the resistance in the secondary circuit. In such a case the galvanometer shows no throw.

If M is the mutual inductance, R is the total resistance in the secondary circuit including the resistance of the secondary coil, G the galvanometer resistance and i the primary current, the charge circulating the galvanometer is $\frac{Mi}{G+R}$. Again if r be the resistance across the capacitance C, the charge flowing to the capacitor is irC, and of this the fraction passing through the galvanometer is $irC.\frac{R}{G+R}$. For a balance in the galvanometer, we should have

$$\frac{Mi}{G+R}=\frac{irCR}{G+R}$$

$$\text{Hence } M=RrC$$

NUMERICAL EXAMPLES

1. A long solenoid having 50 turns per centimetre carries a current of 1 ampere. A metal disc of radius 10 cm. is placed inside the solenoid and is rotated at the rate of 10 revolutions per second, the axis of rotation being the same as the axis of the solenoid. Calculate the potential difference between the centre and periphery of the disc.

Solution : Field inside the solenoid $H=4\pi\times 50\times 0{\cdot}1=20\pi$

Induced *emf* $e=n.\pi r^2H=10\times\pi\times 10^2\times 20\pi$

or $e=2\times 10^4\times\pi^2=19{\cdot}7\times 10^4$ *e.m.u.*

or $e=1{\cdot}97\times 10^{-3}$ volt.

2. A solenoid produces a flux of 3×10^4 maxwells in an iron core for a current of 1·5 amperes through it It has 500 turns. Calculate the self-inductance.

Solution : Total flux linked with 500 turns$=3\times 10^4\times 500$

Flux for *e.m.* unit current$=\frac{3\times 10^4\times 500}{1{\cdot}5\times 10^{-1}}=10^8$

Hence self-inductance$=10^8$ e.m.u.$=0{\cdot}1$ henry

3. An air-core solenoid of 40 cm. length and having 10 turn per centimetre has a diameter of 4 cm. Calculate the self-inductance in henry.

Solution : $L=4\pi n^2 a^2 l$, $n=$ '0, $a=2$ cm. $l=40$ cm.

So $L=4\pi^2.10^2.2^2.10=631{\cdot}014\times10^3$ *e.m.u.*

or $L=631{\cdot}014\times10^3\times10^{-9}=631{\cdot}014\times10^{-6}$ henry

4. Calculate the mutual inductance of two solenoidal coils the inner one having 15 turns per centimetre wound on a wooden cylinder of 100 cm. length and 4 cm. diameter. The outer coil has 100 turns and is closely wound round the central portion of the inner coil.

Solution : $M=\frac{n_s}{n_p}.L$, $n_s=100$, $n_p=100\times15$

$$L=4\pi n_1{}^2 lA=4\pi\times5^2\times100\times\pi\times2^2=38{\cdot}4\times10^4$$

So $M=\frac{100}{1500}\times38{\cdot}4\times10^4=2{\cdot}56\times10^4$ *e.m.u.*

or $M=2{\cdot}56\times10^{-5}$ henry

5. Two coils having $1200\mu H$ and $1800\mu H$ as respective self-inductance are joined so that when the mutual inductance is positive the effective self-inductance of the combination is 4 mH and it is 2 mH when the mutual inductance is negative. Obtain the mutual inductance.

Solution : $L=L_1+L_2+2M=(1200+1800)\mu H+2M=3\ mH+2M$

$L'=L_1+L_2-2M=(1200+1800)\ \mu H-2M=3\ mH-2M$

Hence $L-L'=4M$, and as given $L=4mH$ and $L'=2mH$,

So $4m=(4-2)mH=2\ mH.$

Hence $M=0{\cdot}5\ mH.$

EXERCISES ON CHAPTER V

5·1. Establish mathematically that the induced *emf* in a circuit is equal to the rate of change of flux through it.

5·2. What is Eddy current ? Describe experiments to show its existence and transference of such current to an external circuit.

A circular metal disc of diameter 40 cm. rotates on its own axis 60 times a second, the plane of the disc being normal to a magnetic field of 100 oersteds. If two copper brushes connected through a resistor of 15·7 ohms respectively touch the axis and periphery of the disc, calculate the current through the resistor. [*Ans* : 0·0048 amp.]

5·3. Define coefficient of self-induction. Obtain an expression for the energy stored up in the medium due to a current i established in a coil of self-inductance L.

Calculate the self-inductance of a solenoid and discuss the case where the solenoid is a short one.

A uniform solenoid of length 100 cm. and radius 2 cm. contains 20 turns per centimetre. Calculate the energy dissipated in the spark when the circuit carrying 5 amperes of current is suddenly broken. [*Ans* : $7{\cdot}88\times10^{5}$ ergs.]

5·4. Two parallel wires are lying d distance apart, where $d >> a$, a being the radius of each of the wires. Calculate the self-inductance and discuss its importance when $d \to a$.

Obtain the self-inductance of two co-axial cylinders and indicate the use of such a combination of conductors in the transmission of radio-frequency alternating current.

5·5. Define mutual inductance. Show that the mutual potential energy of two coils having mutual inductance M, each carrying a current is Mi^2. Explain the coefficient of coupling.

A small circular coil of n_1 turns, each of radius a is placed on the axis of and at a distance x from the centre of another coil containing n_2 turns and of same radius. Calculate the mutual inductance of the two coils.

$$\left[Ans : \frac{2\pi^2 a^4 n_1 n_2}{(a^2+x^2)^{\frac{3}{2}}}. \right]$$

5·6. A small coil of 50 turns each of area 100 sq. cm. is placed at the centre of a Helmholtz double coil system each of 100 turns and radius 10 cm. Calculate the mutual inductance. [*Ans* : 44·8 μH]

5·7. Calculate the mutual inductance of two solenoids, one small solenoidal coil wound at the central portion of a longer one.

5·8. Discuss how an interrupted primary current induces current in a closely wound secondary coil.

5·9. Describe the construction and working of the Ruhmkorff's induction coil. Discuss and explain the action of the condenser put in series with the primary coil.

An induction coil has a mutual inductance of 5×10^9 e.m. units. If a current of 5 amperes in the primary collapses completely in 0·01 second, calculate the secondary voltage.

[*Ans* : 2500 volts]

5·10. Describe experimental methods for determination of (a) self-inductance (b) Mutual inductance.

5·11. A solenoid 70 cms. long is wound with 30 turns of wire per centimetre each of radius 4·5 cms. A secondary coil of 750 turns is wound upon the mid-portion of the solenoid. Calculate the self-inductance and mutual inductance of the solenoid. How will the inductance of the solenoid be affected if the secondary coil is closed ?

[*Ans* : $L = 0·05H$, $M = 0·018H$, *Diminished*]

5·12. Calculate the coefficient of self-induction of a solenoid of 400 turns if it produces a flux of 30000 maxwells in an iron core when current of 2 amperes flows through it.

[*Ans* : $L = 0·06\ H$]

5·13. Two solenoids have self-inductances L_1, L_2 and mutual inductance M. For complete linking prove that $M = \sqrt{L_1 L_2}$.

5·14. Give the three different definitions of self-inductance. When do these disagree ?

Define Henry and express it in *e.m.* unit.

CHAPTER VI

VARYING CURRENTS

VI-1. CIRCUITS HAVING INDUCTIVE RESISTANCE

Growth of Current : A coil having a pure resistance R in series with a pure inductance L represents an *inductive circuit.* Let such a combination be employed to join the terminals of a supply of steady *emf* E. As soon as the circuit is closed, a current flows and a flux becomes linked with the coil. This growing flux causes a back *emf* due to electromagnetic induction and it is of magnitude $L\frac{di}{dt}$. Hence the voltage that is available for setting up the current through the resistor is $E-L\frac{di}{dt}$. This *emf* causes a current i through the resistance R, given by the equation

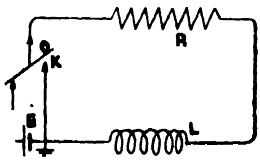


Fig. 6·1

$$E-L\frac{di}{dt} = Ri$$

or $$\frac{di}{dt} = \frac{E-Ri}{L}$$

or $$\frac{d(E-Ri)}{E-Ri} = -\frac{R}{L}dt$$

Integrating, $log_e(E-Ri) = -\frac{R}{L}t+k$, k is a constant. Applying the initial condition, when $t=0$, $i=0$, we get $k=log_e E$.

Hence $$log_e\left[\frac{E-Ri}{E}\right]=-\frac{R}{L}t$$

or $$\frac{E-Ri}{E} = e^{-\frac{R}{L}t}$$

or $$i = \frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)$$

This is the expression for the magnitude of current at any instant. The current instead of attaining the steady value instantaneously from the start grows exponentially and reaches the maximum value when $e^{-\frac{R}{L}t}=0$ *i.e.* when $t \to \infty$. The steady value of the current is given by $i_0=\frac{E}{R}$, hence $i = i_0\left(1-e^{-\frac{t}{\lambda}}\right)$, writing λ for $\frac{L}{R}$ λ is called the *time constant* of the circuit. If we put $t=\lambda$, than $i=i_0\left(1-\frac{1}{e}\right) = 0{\cdot}632 i_0$. So the time constant may be defined as the time in which the current reaches 63 percent of its maximum value. Though theoretically the current should attain its maximum value in infinitely long time, in practice it is obtained within a time which is not very long. Iron-cored coils for which L is comparatively large takes several seconds to reach its maximum value. The nature of growth of current as determined by different values of λ is shown in curves drawn in Fig. 6·2.

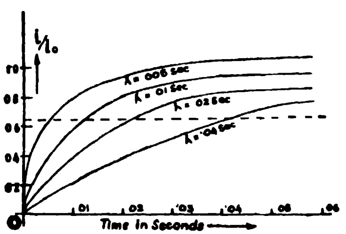


Fig. 6·2

Decay of Current: Let a circuit containing a resistance R and an inductance L be completed through a battery of *emf* E and let it produce a current i. If now the key is thrown off the battery and made to form a closed circuit containing the resistance and and the inductance the differential equation for the current in the closed circuit becomes

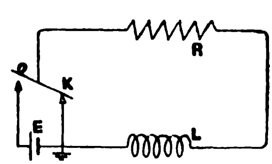


Fig. 6·3

$$L\frac{di}{dt}+Ri=0$$

or
$$\frac{di}{i}=-\frac{R}{L}.dt$$

Integrating, $\log_e i = -\frac{R}{L}t + k$, k is a constant. Initially when $t=0$, $i=i_0=\frac{E}{R}$, the steady current, hence $k=\log_e i_0$

Therefore $\log_e \left(\frac{i}{i_0}\right) = -\frac{R}{L}t$

$$\text{or} \qquad i = i_0 e^{-\frac{R}{L}t} = \frac{E}{R} e^{-\frac{R}{L}t}$$

Though there is no battery in the circuit, the current persists for sometime and decays exponentially.

Putting $\lambda=\frac{L}{R}$, $i=i_0 e^{-\frac{t}{\lambda}}$.

when $t=\lambda$, $\frac{i_0}{e}=0\ 308 i_0$

The time-constant λ may be considered as equal to the time for the current to decay by 63 percent of its initial maximum value.

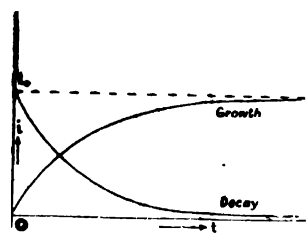


Fig. 6·4

It may be noted that growing and decaying currents are complimentary (Fig. 6·4).

VI-2. CAPACITANCE AND RESISTANCE CIRCUIT

Charging of a condenser : Let the terminals of a condenser having a capacitance C be connected through a resistance R to the terminals of a battery of *emf E*. As soon as the circuit is closed charge flows from the battery to the condenser and this flow of charge through the resistance constitutes a current (i). The equation for the charge in the condenser at any instant may be written as shown overleaf considering the potential drop at the terminals of the resistor.

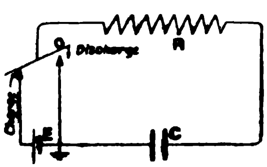


Fig. 6·5

$$\frac{Q}{C}=E-Ri$$

$$\text{or} \quad Ri+\frac{Q}{C}=E$$

$$\text{Since } i=\frac{dQ}{dt},\ R\frac{dQ}{dt}+\frac{Q}{C}=E$$

$$\text{or} \quad \frac{dQ}{dt}+\frac{Q}{C/R}=\frac{E}{R}$$

$$\text{or} \quad \frac{dQ}{dt}=\frac{1}{R}(E-Q/C)$$

$$\text{or} \quad \frac{d(E-Q/C)}{(E-Q/C)}=-\frac{dt}{CR}$$

$$\text{Integrating, } \log_e\left(E-\frac{Q}{C}\right)=-\frac{t}{CR}+K$$

Initially when $t=0$, $Q=0$, hence $K=\log_e E$

$$\text{Therefore, } \log_e\left[\frac{E-Q/C}{E}\right]=-\frac{t}{CR}$$

$$\text{or} \quad E-\frac{Q}{C}=Ee^{-\frac{t}{CR}}$$

$$\text{or} \quad Q=EC\left(1-e^{-\frac{t}{CR}}\right)$$

When $t\rightarrow\infty$, $Q=EC=Q_0$, the maximum charge

$$\text{or} \quad Q=Q_0\left(1-e^{-\frac{t}{CR}}\right)$$

The charge grows exponentially (Fig. 6·6) being delayed due to the action of the resistance. If λ be written for CR, $Q=Q_0\left(1-e^{-\frac{t}{\lambda}}\right)$. λ is called the *time constant*. When $\lambda=t$, $Q=0{\cdot}632Q_0$. Hence the time constant is the time required for the capacitor to acquire 63 percent of its maximum charge.

$$i=\frac{dQ}{dt}=\frac{Q_0}{CR}e^{-\frac{t}{\lambda}}=\frac{E}{R}e^{-\frac{t}{\lambda}}=i_0e^{-\frac{t}{\lambda}},$$

Where i_0 is $=\frac{E}{R}$, the initial maximum current.

Considering the instantaneous voltage V given by $\frac{Q}{C}$ at the terminals of the capacitor, the equation $Q=EC\left(1-e^{-\frac{t}{\lambda}}\right)$ may be written as

$$V=E\left(1-e^{-\frac{t}{\lambda}}\right)$$

Discharge of a condenser : When a charged condenser is short-circuited by disconnecting the battery (Fig. 6·5) the condenser begins to discharge. If Q be the charge on the condenser of capacitance C at any instant t after the removal of the battery, the potential of the condenser is Q/C. If i be the current the potential drop at the ends of the resistance R is Ri. These two balance one another and hence,

$$Ri+\frac{Q}{C}=0$$

$$\text{or} \quad R\frac{dQ}{dt}+\frac{Q}{C}=0$$

$$\text{or} \quad \frac{dQ}{dt}=-\frac{Q}{CR}$$

$$\text{or} \quad \frac{dQ}{Q}=-\frac{dt}{CR}$$

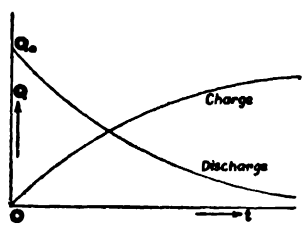


Fig. 6·6

Integrating, $\log_e Q=-\frac{t}{CR}+k$, k being a constant.

When $t=0$, the initial charge $Q=Q_0=EC$, so

$k=\log_e Q_0$. Hence

$$\log_e\left(\frac{Q}{Q_0}\right)=-\frac{t}{CR}$$

$$\text{or} \quad Q=Q_0 e^{-\frac{t}{CR}}=ECe^{-\frac{t}{CR}}$$

The discharge is not instantaneous but the charge of the condenser falls exponentially (Fig. 6·6) and is delayed by the action of the resistance R.

Putting $\lambda=\frac{1}{CR}$, $Q=Q_0 e^{-\frac{t}{\lambda}}$, λ is called the *time constant.*

When $t=\lambda$, $Q=0{\cdot}368\ Q_0$. Time constant is the interval necessary for the condenser to lose 63 percent of its charge.

$$\text{Current } i=\frac{dQ}{dt}=-\frac{Q_0}{CR}\,e^{-\frac{t}{CR}}=-\frac{E}{R}e^{-\frac{t}{CR}}=-i_0e^{-\frac{t}{CR}}$$

where i_0 is the initial current through R. The potential of the condenser at any instant is obtained as

$$V=\frac{Q}{C}=\frac{Q_0}{C}\cdot e^{-\frac{t}{\lambda}}=Ee^{-\frac{t}{\lambda}}$$

Both in the process of charging and discharge the current starts with its maximum value and decays exponentially to zero, but the direction of the current during discharge is opposite to that at the time of charging.

VI-3. OSCILLATORY CIRCUITS

Charging of a condenser through inductance : Suppose a battery of *emf* E is connected to a condenser of capacitance C through an inductance L. As soon as the circuit is closed charge passes to the capacitor causing a current through the inductance and a back *emf* of magnitude $L\frac{di}{dt}$ is induced. It opposes the *emf* of the battery. Hence at any instant the potential difference at the ends of the capacitor in terms of its charge and capacitance is given by

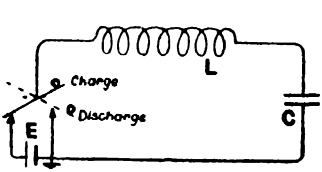


Fig. 6·7

$$\frac{Q}{C}=E-L\frac{di}{dt}=E-L\frac{d^2Q}{dt^2}$$

$$\text{or } L\frac{d^2Q}{dt^2}=E-\frac{Q}{C}=\frac{EC-Q}{C}$$

$$\text{or } \frac{d^2Q}{dt^2}=\frac{EC-Q}{LC}$$

Putting $Q-EC=x$, $\dfrac{d^2Q}{dt^2}=\dfrac{d^2(Q-EC)}{dt^2}=\dfrac{d^2x}{dt^2}$

$$\text{Hence } \frac{d^2x}{dt^2}=-\frac{1}{LC}.x.$$

Again putting $\omega^2 = \frac{1}{LC}$, $\frac{d^2x}{dt^2} = -\omega^2 x$.

Let $x = A_0 e^{pt}$ be a solution of the equation.

Hence $\frac{d^2x}{dt^2} = p^2 A_0 e^{pt} = p^2 x$

Putting this value of $\frac{d^2x}{dt^2}$ in the differential equation

we get, $p^2 + \omega^2 = 0$

or $p = \pm j\omega$, where $j = \sqrt{-1}$

So $x = A_1 e^{j\omega t} + A_2 e^{-j\omega t}$, A_1 and A_2 being constants.

or $x = A_1(\cos\omega t + j\sin\omega t) + A_2(\cos\omega t - j\sin\omega t)$

or $x = (A_1 + A_2)\cos\omega t + (A_1 - A_2)\, j\sin\omega t$

or $x = A\cos\omega t + B\sin\omega t$, A and B are new constants.

Substituting for x its value $Q - EC$, we get

$$Q - EC = A\cos\omega t + B\sin\omega t$$

At the start when $t = 0$, $Q = 0$, hence $A = -EC$

Also when $t = 0$, $\frac{dQ}{dt} = 0$, hence $B = 0$

so $Q - EC = -EC\cos\omega t$

or $Q = EC(1 - \cos\omega t) = Q_0(1 - \cos\omega t)$

Q_0 is the steady charge EC that would have been obtained by direct contact of the battery with the condenser. The charge varies in a *sinusoidal manner*, as shown in the graph (Fig. 6·8), on either side of the steady charge.

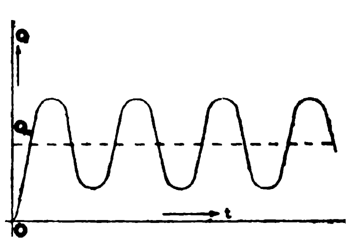


Fig. 6·8

Current $i = \frac{dQ}{dt} = -Q_0\,\omega\sin\omega t$

or $i = -\frac{Q_0}{\sqrt{LC}}\sin\omega t = i_0\sin\omega t$,

writing $-\frac{Q_0}{\sqrt{LC}} = i_0$

The current also is simple harmonic in nature, the frequency being given by $f = \frac{\omega}{2\pi}$ i.e. $f = \frac{1}{2\pi\sqrt{LC}}$

The charge of the condenser surges backward and for-

ward through the inductance being alternately maximum on either plate. At an instant halfway between the two extremes, the charge upon the plate is Q_0.

This is an ideal case, never realised in practice. We have assumed the circuit to possess no ohmic resistance, which is never obtained.

Discharge of a condenser through an inductance : Let a charged condenser of capacitance C having a charge Q_0 be short-circuited through an inductance L. The passage of the charge causes a back *emf* $-L\frac{di}{dt}$ which tends to drive the charge in the opposite direction. The potential difference between the plates of the condenser at any instant is Q/C and hence the equation for the discharge may be written as

$$-L\frac{di}{dt} = \frac{Q}{C}$$

$$\text{or} \quad \frac{d^2Q}{dt^2} = -\frac{Q}{LC} = -\omega^2 Q, \text{ putting } \frac{1}{LC} = \omega^2$$

This equation may be solved in the same manner as shown in the foregoing section putting $Q=A_0e^{pt}$ as a solution. The value of Q is obtained as $Q=Q_0 \cos \omega t$, where $Q_0=EC$, the steady charge that would have been obtained by connecting the battery directly with the condenser or the maximum charge of the condenser before short-circuiting it. The discharge is oscillatory, the frequency of oscillation being given by $f=\omega/2\pi=1/2\pi\sqrt{LC}$. The charge surges backwards and forwards through the inductance being alternately positive and negative upon either plate of the condenser. For either extreme the energy of charge is $\frac{Q_0^2}{2C}$ and at an instant halfway between the extremes, the charge upon either of the plates is zero, and the energy is wholly associated with the inductance having the value $\frac{1}{2}Li_0^2$.

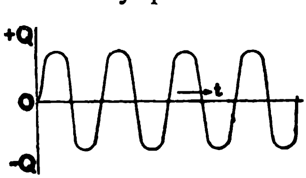


Fig. 6·9

$$\text{The current } i=\frac{dQ}{dt}$$

$$\text{or } i=-\omega Q_0 \sin \omega t$$

$$\text{or } i=i_0 \sin \omega t, \; i_0=-\omega Q_0.$$

Current of this type is known as *alternating current.* The production of sinusoidal current in this way has important applications. By this method we may generate alternating current of any desired frequency by proper choice of the values of L and C. High frequency current as necessary in radio-transmission is obtained by this principle of discharge of a condenser through an inductance.

VI-4. DAMPED OSCILLATORY CIRCUITS

Charging a condenser through an inductive resistance : If an *emf* (E) is applied to a circuit containing a resistance (R), an inductance (L) and a capacitance (C) in series, the *emf* of the battery available for sending a current through the resistance is opposed by the induced *emf* in the inductance and the potential difference at the plates of the condenser. Hence if i be the instantaneous current through the resistance and Q be the charge in the condenser at any instant, we have,

$$Ri=E-\left(L\frac{di}{dt}+\frac{Q}{C}\right)$$

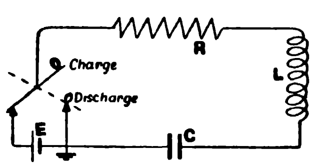


Fig. 6·10

$$\text{or } L\frac{di}{dt}+Ri+\frac{Q}{C}=E$$

$$\text{or } L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q-EC}{C}=0$$

$$\text{or } \frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q-EC}{LC}=0$$

Put $x=Q-EC=Q-Q_0$, Q_0 being the steady charge.

$$\text{Hence } \frac{dx}{dt}=\frac{dQ}{dt}, \quad \frac{d^2x}{dt^2}=\frac{d^2Q}{dt^2},$$

Putting $\frac{R}{L}=2k$ and $\frac{1}{LC}=\omega^2$, the equation may be written as,

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+\omega^2x=0$$

Let $x = Pe^{mt}$ be a solution of the equation, hence $\frac{dx}{dt} = mPe^{mt} = mx$ and $\frac{d^2x}{dt^2} = m^2Pe^{mt} = m^2x$.

So the equation is obtained as

$$m^2 + 2km + \omega^2 = 0$$

It m_1 and m_2 be the roots of this equation

$m_1 = -k + \sqrt{k^2 - \omega^2}$ and $m_2 = -k - \sqrt{k^2 - \omega^2}$,

Hence $m_1 - m_2 = 2\sqrt{k^2 - w^2} = 2b$ (say)

So $m_1 = -k + b$ and $m_2 = -k - b$, also $m_1 m_2 = \omega^2$

Thus writing the solution in terms of the roots,

$$x = Ae^{m_1 t} + Be^{m_2 t}$$

$$\frac{dx}{dt} = m_1 Ae^{m_1 t} + m_2 Be^{m_2 t}$$

When $t=0$, $x = A+B$ so $Q - Q_0 = A + B$, but since when $t=0$, $Q=0$, so $A+B = -Q_0$.

Again when $t=0$, $\frac{dx}{dt} = m_1 A + m_2 B = 0$

Hence $A = Q_0 \frac{m_2}{m_1 - m_2}$

and $B = -Q_0 \frac{m_1}{m_1 - m_2}$

So $x = \frac{Q_0}{m_1 - m_2}\left[m_2 e^{m_1 t} - m_1 e^{m_2 t} \right]$

or $x = \frac{Q_0}{2b}\left[-(k+b)\, e^{(-k+b)t} - (-k+b)\, e^{-(k+b)t} \right]$

or $Q - Q_0 = \frac{-Q_0 e^{-kt}}{2b}\left[-(k+b)\, e^{bt} - (-k+b)\, e^{-bt} \right]$

or $Q = Q_0\left[1 - \frac{e^{-kt}}{2b} \left\{ (k+b)\, e^{bt} - (k-b)\, e^{-bt} \right\} \right]$

The charge gradually approaches the steady value Q_0, the mode of approach is somewhat exponential.

OSCILLATORY CHARGING : If the components of the circuit be such that $k^2 < \omega^2$, then $b = \sqrt{k^2 - \omega^2}$ becomes ima-

ginary and so b may be written as jp, where $p^2=\omega^2-k^2$, and so the equation for Q should be written as

$$Q=Q_0\left[1-\frac{e^{-kt}}{2jp}\left\{(k+jp)e^{jpt}-(k-jp)e^{-jpt}\right\}\right]$$

$$\text{or } Q=Q_0\left[1-\frac{e^{-kt}}{2jp}\left\{k(e^{jpt}-e^{-jpt})+jp(e^{jpt}+e^{-jpt})\right\}\right]$$

$$\text{or } Q=Q_0\left[1-\frac{e^{-kt}}{2jp}\left\{2jk\sin pt+2jp\cos pt\right\}\right]$$

$$\text{or } Q=Q_0\left[1-\frac{e^{-kt}}{2p}(k\sin pt+p\cos pt)\right]$$

Put $k=a\sin\phi$ and $p=a\cos\phi$, so $\tan\phi=k/p$ and $a^2=k^2+p^2=k^2-k^2+\omega^2=\omega^2$, or $a=\omega$, $k=\omega\sin\phi$ and $p=\omega\cos\phi$.

$$\text{Therefore } Q=Q_0\left[1-\frac{e^{-kt}}{p}(\omega\sin pt\sin\phi+\omega\cos pt\cos\phi\right]$$

$$\text{or } Q=Q_0\left[1-\frac{e^{-kt}}{p}\omega\cos(pt-\phi)\right]$$

$$\text{or } Q=Q_0\left[1-\frac{\omega e^{-kt}}{\sqrt{\omega^2-k^2}}\cos(\sqrt{\omega^2-k^2}.\,t-\phi)\right]$$

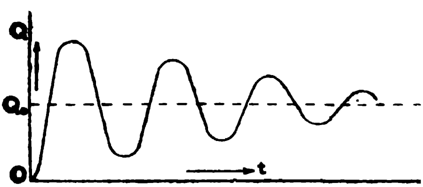


Fig. 6·11

The charge varies in oscillatory character and it gradually loses amplitude. The amplitude initially may be many times greater than the final steady value. The frequency for oscillation is given by

$$f=\frac{\sqrt{\omega^2-k^2}}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$$

Expression for the current : Current i is obtained as $\frac{dQ}{dt}$ and so

$$=\frac{d}{dt}\left[Q_0\left\{1-\frac{\omega e^{-kt}}{p}\cos(pt-\phi)\right\}\right]$$

$$\text{or } i=Q_0\left[\omega e^{-kt}\sin(pt-\phi)+\frac{\omega k e^{-kt}}{p}\cos(pt-\phi)\right]$$

or $$i = \frac{Q_0 e^{-kt}\omega}{p}\Big[p \sin(pt-\phi) + k\cos(pt-\phi)\Big]$$

or $$i = \frac{Q_0 e^{-kt}\omega}{p}\Big[p \sin pt \cos\phi - p\cos pt \sin\phi$$
$$+ k\cos pt\cos\phi + k\sin pt \sin\phi\Big]$$

Since $k = p\tan\phi$, hence by substitution for k, we get

$$i = \frac{Q_0 e^{-kt}\omega}{p}\Big[p \sin pt\cos\phi - p\cos pt\sin\phi$$
$$+ p\cos pt\sin\phi + \frac{p\sin^2\phi \sin pt}{\cos\phi}\Big]$$

or $$i = \frac{Q_0 e^{-kt}\omega}{p}\Big[p\sin pt\cos\phi + p\sin pt\frac{\sin^2\phi}{\cos\phi}\Big]$$

or $$i = \frac{Q_0 e^{-kt}\omega}{p}\Big[p\sin pt(\cos^2\phi + \sin^2\phi)\frac{1}{\cos\phi}\Big]$$

or $$i = \frac{Q_0 e^{-kt}\omega}{p}\, p\sin pt\,\frac{\omega}{p}, \text{ since } \cos\phi = \frac{p}{\omega}$$

or $$i = Q_0\frac{e^{-kt}.\omega^2}{p}\sin pt$$

or $$i = \frac{Q_0 e^{-\frac{R}{2L}t}}{LC\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\sin\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,.\,t$$

The current is oscillatory with gradually diminishing amplitude. The frequency of oscillation is

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$$

Discharge of condenser through inductive resistance : A condenser of capacitance C and charge Q_0 is short-circuited through an inductance L and resistance R (Fig. 6·10). The discharge causes a gradual fall of charge, indicated as $-\frac{dQ}{dt}$ in the capacitor and this sends a current through the resistor R, given by $i = -\frac{dQ}{dt}$.

Equating the ohmic potential drop at the ends of the resistor with that at the plates of the condenser, having a a charge Q, diminished by the back *emf* in the inductance, we may write

$$Ri=\frac{Q}{C}-L\frac{di}{dt}$$

or $$-R\frac{dQ}{dt}=\frac{Q}{C}+L\frac{d^2Q}{dt^2}$$

or $$\frac{d^2Q}{dt^2}+\frac{R}{L}.\frac{dQ}{dt}+\frac{Q}{LC}=0$$

Putting $\frac{R}{L}=2k$, and $\frac{1}{LC}=\omega^2$, we may write

$$\frac{d^2Q}{dt^2}+2k\frac{dQ}{dt}+\omega^2Q=0$$

Let $Q=Pe^{mt}$ be a solution and hence $\frac{dQ}{dt}=mQ$ and $\frac{d^2Q}{dt^2}=m^2Q$, so the equation stands as

$$m^2+2km+\omega^2=0$$

If m_1 and m_2 be the two roots of the equation $m_1=-k+\sqrt{k^2-\omega^2}$ and $m_2=-k-\sqrt{k^2-\omega^2}$, $m_1m_2=\omega^2$ and $m_1-m_2=2\sqrt{k^2-\omega^2}=2b$ (say), hence $m_1=-k+$b and $m_2=-k-b$.

Thus the value of Q may be expressed as

$$Q=Ae^{m_1t}+Be^{m_2t}$$

when $t=0$, $Q=Q_o$, so $A+B=Q_o$

Again when $t=0$, $\frac{dQ}{dt}=0$ so $m_1A+m_2B=0$

Hence $A=\frac{-Q_0m_2}{m_1-m_2}$ and $B=\frac{Q_0m_1}{m_1-m_2}$

So $$Q=\frac{Q_0}{m_1-m_2}\left[-m_2e^{m_1t}+m_1e^{m_2t}\right]$$

or $$Q=\frac{Q_0}{2b}\left[(k+b)e^{(-k+b)t}-(k-b)e^{(-k-b)t}\right]$$

or $$Q=\frac{Q_0e^{-kt}}{2b}\left[(k+b)e^{bt}-(k-b)e^{-bt}\right]$$

The charge of the condenser decreases exponentially with time and reaches zero when $t \to \infty$. The discharge may be stated as dead beat when the decrease is rapid which depends upon the value of k, *i.e.*, $R/2L$.

OSCILLATORY DISCHARGE : If $k^2 < \omega^2$, $\sqrt{k^2-\omega^2}$ becomes imaginary. Writing $\omega^2 - k^2 = p$, we get $b = jp$.

Hence $$Q = \frac{Q_0 e^{-kt}}{2\,jp}\left[\,(k+jp)e^{jpt} - (k-jp)e^{-jpt}\,\right]$$

or $$Q = \frac{Q_0 e^{-kt}}{2jp}\left[\,k(e^{jpt} - e^{-jpt}) + jp(e^{jpt} + e^{-jpt})\,\right]$$

or $$Q = \frac{Q_0 e^{-kt}}{2\,jp}\left[\,2jk \;\sin pt + 2jp \cos pt\,\right]$$

or $$Q = \frac{Q_0 e^{-kt}}{p}\left[\,k \;\sin\; pt + p \;\cos\; pt\,\right]$$

If we write $k = a \sin\phi$ and $p = a \cos\phi$, $\tan\phi = k/p$

and $a^2 = k^2 + p^2 = k^2 + \omega^2 - k^2 = \omega^2$

or $a = \omega$ and $k = \omega \sin\phi$, $p = \omega \cos\phi$

Therefore $$Q = \frac{Q_0 e^{-kt}}{p}\left[\,\omega \sin pt \sin\phi + \omega \cos pt \cos\phi\,\right]$$

or $$Q = \frac{Q_0 e^{-kt}}{p} \cos(pt - \phi)$$

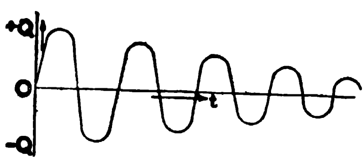


Fig. 6·12

The discharge is oscillatory and the amplitude of the oscillation, that is the maximum value of Q diminishes gradually according to the factor e^{-kt}. The frequency of oscillation is given by $f = p/2\pi$, where $p = \sqrt{\omega^2 - k^2}$, $\omega^2 = 1/LC$ and $k = R/2L$, hence,

$$f = \frac{\sqrt{\omega^2 - k^2}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

When $R \to 0$, $f = \dfrac{1}{2\pi\sqrt{LC}}$.

Expression for the current : We have obtained for the charge in the condenser at any instant, the equation

$$Q = \frac{Q_0 e^{-kt}\omega}{p} \cos(pt-\phi)$$

Current $i=\frac{dQ}{dt}=\frac{Q_0\omega}{p}\cdot\frac{d}{dt}\left[e^{-kt}\cos(pt-\phi)\right]$

or $i=\frac{Q_0\omega}{p}\left[-ke^{-kt}\cos(pt-\phi)-e^{-kt}.\,p\sin(pt-\phi)\right]$

or $i=-\frac{Q_0\omega e^{-kt}}{p}\left[k\cos(pt-\phi)+p\sin(pt-\phi)\right]$

Substituting $p\tan\phi$ for k, as shown in the previous section, we get

$$i=-\frac{Q_0\omega^2 e^{-kt}}{p}\sin pt$$

or $i=-\frac{Q_0 e^{-\frac{R}{2L}t}}{LC\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\sin\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,.\,t$

When $R\to 0$, the oscillations become undamped and the current may be expressed as

$$i=\frac{Q_0}{\sqrt{LC}}\sin\frac{t}{\sqrt{LC}}$$

Frequency of undamped oscillation is $f=\frac{1}{2\pi\sqrt{LC}}$. This particular value has been obtained independently in § VI-3.

CRITICAL DAMPING : If $k^2=\omega^2$, the discharge reaches a critical stage and the damping is such that deat beat discharge becomes most rapid. For a slight increase of ω, $k^2<\omega^2$ and the discharge, as shown, becomes oscillatory. This condition is obtained when $\frac{R^2}{4L^2}=\frac{1}{LC}$ *i.e.* when $R^2=\frac{4L}{C}$. Thus the value of R determines the nature of discharge.

VI-6. MEASUREMENT OF HIGH RESISTANCE

Method of leakage : If a condenser of capacitance C and initial charge Q is allowed to discharge through a high resis-

tance R, the charge Q_0 remaining in the condenser after a time t is given by (§ VI-2),

$$\log_e\left(\frac{Q_0}{Q}\right)=\frac{t}{CR}$$

$$\text{Hence } R=\frac{t}{C\log_e\left(\frac{Q_0}{Q}\right)}$$

R may be obtained by determining the quantity Q/Q_0. A ballistic galvanometer may be used for this purpose in an arrangement shown in Fig. 6·13.

A condenser of capacitance C is charged fully (Q_0) and allowed to discharge instantaneously through the ballistic galvanometer. The throw θ_0 is noted. It is again charged and allowed to leak for t seconds through the high resistance R. Then it is discharged through the galvanometer. The throw θ now obtained is proportional to the charge Q remaining in the condenser after discharge for t seconds. Hence

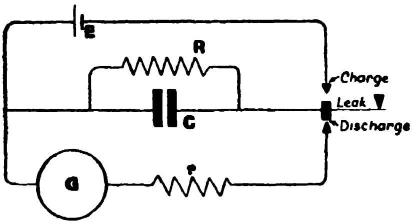


Fig. 6·13

$$R=\frac{t}{C\log_e\left(\frac{Q_0}{Q}\right)}$$

$$=\frac{t}{C\log_e\left(\frac{\theta_0}{\theta}\right)}$$

If C is of the order of 10^{-6} farad, the resistance of the order of 10 meg-ohms and upwards can be measured by the method.

NUMERICAL EXAMPLES

1. A telephone is operated by a battery of 24 volts, having a negligible resistance. The telephone has an inductance of 10 henries and resistance 100 ohms. If the operating current is 120 mA, calculate the operating time after the voltage is applied.

Solution : $i=i_0\left(1-e^{-\frac{Rt}{L}}\right)$ or $\frac{i}{i_0}=1-e^{-\frac{Rt}{L}}$

So $1-\frac{i}{i_0}=e^{-\frac{Rt}{L}}$, hence $-\frac{R}{L}t=log_e\left(1-\frac{i}{i_0}\right)$

$i_0=\frac{24}{100}=0{\cdot}24$ amp and $i=120\ mA=0{\cdot}12$ amp.

So, $-\frac{R}{L}t=log_e\left(1-\frac{{\cdot}12}{{\cdot}24}\right)=-\ log_e 2$

or $\frac{100}{10}t=2{\cdot}303\times{\cdot}3010$, hence $t=0{\cdot}069$ sec.

2. *Calculate the value of the current after 0·05 seconds, if a p.d. of 100 volts applied to a circuit of resistance 50 ohms and self-inductance 5 henries is suddenly withdrawn, the circuit being shorted immediately.*

Solution : $i_0=\frac{100}{50}=2\ amps.$

$$log_e\left(\frac{i_0}{i}\right)=\frac{R}{L}t=\frac{50\times0{\cdot}05}{5}=0{\cdot}5$$

Putting $\frac{i_0}{i}=x$,

$2{\cdot}303\ log_{10}x=0{\cdot}5$

or $log_{10}x=\frac{0{\cdot}5}{2{\cdot}303}=0{\cdot}2172$

So $\frac{i_0}{i}=x=1{\cdot}65$

hence $i=\frac{i_0}{1{\cdot}65}=\frac{2}{1{\cdot}65}=1{\cdot}21$ amp.

3. *A charged condenser of 1μF capacitance is short-circuited through a resistance so that the charge is reduced to 50 p.c in 6·93 seconds. Calculate the value of the resistance.*

Solution : $\frac{Q_0}{Q}=2,\ log_e\left(\frac{Q_0}{Q}\right)=\frac{t}{CR}$

So $2{\cdot}303\ log_{10}2=\frac{6{\cdot}93}{10^{-6}\times R}$

or $R=\frac{6{\cdot}93\times10^6}{2{\cdot}303\times0{\cdot}3010}=10$ meg-ohms.

4. (*a*) *Obtain the circular frequency of oscillation of a circuit formed with a condenser of capacitance $10^{-2}\mu F$ and a coil of self-inductance 0·4 mH. Neglect the resistance of the coil.*

Solution : $p=2\pi f=\dfrac{1}{\sqrt{LC}}=\dfrac{1}{\sqrt{0{\cdot}4\times10^{-3}\times10^{-8}}}=\dfrac{1}{2\times10^{-6}}$

or $p=0{\cdot}5\times10^{6}=500\times10^{3}=500$ kilocycles/sec.

(*b*) *In the foregoing problem obtain the frequency if the resistance in the circuit is 80 ohms.*

Solution : $f=\dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC}-\dfrac{R^2}{4L^2}}$

$$=\frac{1}{2\pi}\sqrt{\frac{1}{0{\cdot}4\times10^{-11}}-\frac{80^2}{4(0{\cdot}4\times10^{-3})^2}}$$

or $f=\dfrac{1}{2\pi}\sqrt{25\times10^{10}-10^{10}}=\dfrac{10^5\times4{\cdot}9}{2\pi}$

or $f=77{\cdot}98$ kilocycles per second.

5. *A condenser with a capacitance of $25\mu F$ is discharged through a coil of self-inductance 4 henries. What resistance should be included in the circuit so that the discharge may be critically damped ?*

Solution : For critical damping $R^2=\dfrac{4L}{C}$

So $R^2=\dfrac{4\times4}{25\times10^{-6}}=\dfrac{4\times4\times10^{6}}{25}$

or $R=0{\cdot}8\times10^{3}=800$ ohms.

6(*a*). *Calculate the leakage resistance of a condenser of capacitance $6\mu F$ which on being charged loses 90 percent of its charge in 10 minutes.*

Solution : $\log_e\left(\dfrac{Q_0}{Q}\right)=\dfrac{t}{CR}$

or $2{\cdot}303\times\log_{10}\left(\dfrac{100}{90}\right)=\dfrac{10\times60}{6\times10^{-6}\times R}$ ... ... (i)

or $(2{\cdot}303\times0{\cdot}0458)R=10^{8}$

Hence $R=948$ meg-ohms.

(*b*). *A resistance is joined across the condenser and It is found to lose 50 p.c. of its charge in the same interval. Calculate the value of this resistance.*

Solution : $$log_e\left(\frac{Q_o}{Q}\right)=log_e 2=\frac{10\times 60(R+r)}{6\times 10^{-6}\times Rr} \quad \cdots \quad \cdots \quad \text{(ii)}$$

From (i) and (ii) $$\frac{0{\cdot}0458}{0{\cdot}3010}=\frac{r}{948+r}$$

Hence $r=153{\cdot}2$ meg-ohms.

EXERCISES ON CHAPTER VI

6·1. Discuss the growth of current in a coil having inductance and resistance when a source of constant *emf* is applied to the circuit. What is time constant ?

Show that the decay of current in the same circuit when the *emf* is withdrawn is complimentary to the growth.

6·2. Discuss the nature of discharge of a condenser through a resistance. What is time constant ?

A condenser of capacitance $0{\cdot}1\mu F$ is charged and when it is allowed to leak through a resistance its potential falls to 25 p.c. of its original value in 0·693 second. Calculate the resistance.

[*Ans* : 5 meg-ohm]

6·3. Investigate the nature of charging a condenser through a resistance.

A $25\mu F$ capacitor is in series with a 1000 ohms resistance and a 200-volt source is applied to it. Determine (i) the initial current (ii) the time constant (iii) the charge when the time is equal to the time constant and (iv) energy stored up in the capacitor at that instant.

[*Ans* : 0·2 amp., 0·025 sec., 0·00316 coulomb, 0·199 joule]

6·4. Find an expression for the current at any instant in a circuit containing a capacitance, an inductance and resistance connected in series, when a source of constant *emf* is applied. Obtain the condition for the current to be oscillatory.

6·5. A charged capacitor of capacitance C is discharged through a resistance and inductance. Obtain the nature of discharge and indicate when it is oscillatory.

What is the resistance of the circuit with a capacitor $2\mu F$ and inductance 0·5 henry when it becomes critically damped ?

[*Ans* : 1000 ohms]

6·6. Explain the theory underlying the measurement of a high resistance by method of discharge of condenser through it. Describe the experimental arrangement.

A condenser of capacitance $10\mu F$ is charged to a potential of 200 volts. On short-circuiting it through a high resistance the voltage falls to 120 volts in 40 seconds. Calculate the value of the high resistance.

[*Ans* : $7{\cdot}83 \times 10^6$ ohms]

6·7. A condenser of capacitance C is charged to a potential E and discharged through a coil of resistance R and self-inductance L. Describe the condition for producing an oscillatory discharge. How can you demonstrate oscillatory nature of discharge ?

6·8. A charged condenser is discharged through an inductance of negligible resistance. Discuss what happens and state the importance of such a circuit.

Determine the frequency of oscillations in the case of a Leyden jar of $0{\cdot}001\mu F$ capacitance discharged through an inductance of 0·004 henry.

[*Ans* : 79·56 kcs per sec]

6·9. A leaky condenser loses 10 p.c. of its charge in 5 minutes. If the terminals are short-circuited through a resistance of 10 meg-ohms, the loss of voltage is 50 p.c. in the same interval. Calculate the resistance of the insulation.

[*Ans* : 0·18 meg-ohm]

6·10. A capacitance of $0{\cdot}16\mu F$ is discharged through an inductance of 0·4 henry. What resistance added in the circuit makes the discharge just non-oscillatory ?

[*Ans* : 10 meg-ohms.]

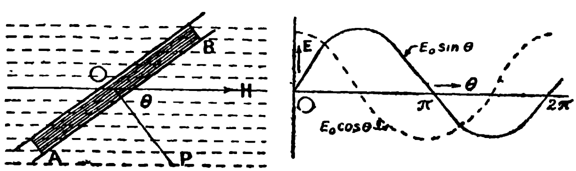
CHAPTER VII

A. C. CIRCUITS

VII-1. SINUSOIDAL E. M. F.

Induced current in a rotating coil: If a closed coil be rotated in a uniform magnetic field, a current flows through it due to induced *emf* caused by change of flux linked with the circuit.

Let a coil having n turns each of area A be rotated with angular velocity ω in a uniform magnetic field of intensity H. Let at any instant t the normal to the coil make an angle



(a) Fig. 7·1 (b)

$\theta=\omega t$ with the direction of the field. The flux linked with the coil in this position is given by $N=AnH\cos\theta$. Hence as the coil rotates the change of flux produced generates an *emf* in the coil given by

$$E= -\frac{dN}{dt}= -\frac{d}{dt}(AnH\cos\theta)= -\frac{d}{dt}(AnH \cos \omega t)$$

or $E= -AnH\omega \sin \omega t = E_0 \sin \omega t = E_0 \sin \theta$.

E indicates the instantaneous *emf* at an instant denoted by t or at any position θ of the normal to the coil with respect to the field. The *emf* is sinusoidal, the amplitude or the peak value of the *emf* being $E_0 = AnH\omega$. θ is the phase angle.

Such an *emf* is called an alternating *emf* and the associated current in the coil expressed in the form of $i_0 \sin\theta$ is an ***alternating current***. The frequency of the *emf* is given by $f=\omega/2\pi$, the period of oscillation is $T=2\pi/\omega$. ω is called the ***circular frequency*** or ***pulsatance***. It is denoted also by p.

If ϕ is considered as the angle made by the *plane of the coil* with the field then the flux at any instant is $N = AnH \sin\phi$ and the induced *emf* is obtained as a cosine function *i.e.* $AnH\omega\cos\phi = E_0 \cos\phi$. Alternatively, if at the time of start the normal to the coil be making an angle $\frac{\pi}{2}$ with the field *i.e.* when $t=0$, $\theta = \frac{\pi}{2}$, so $\omega t = \left(\theta - \frac{\pi}{2}\right)$, so $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right) = \cos \omega t$. Therefore the *emf* $E = E_0 \sin\theta = E_0 \cos \omega t$. In other words, if the time t is reckoned from the instant or position of the coil when the *emf* is maximum in the coil, the equation for the *emf* is $E = E_0 \cos \omega t$. On the other hand if this is counted from the position or instant of the zero *emf*, the equation should be taken as $E = E_0 \sin \omega t$.

The wave form of an alternating current instead of being sinusoidal may be a complex one. Our discussion will be limited to the sinusoidal *emf* and current only.

Earth inductor : The earth inductor is a contrivance for generation of sinusoidal *emf* by induction caused by earth's field. It is a rectangular or circular coil consisting of several turns of insulated wire mounted on an axle so that it can be rotated at any angle with the earth's magnetic field. When the coil is rotated there is a change of flux through it. The current set up in the coil may be conveyed to any external circuit by means of a ring-and-brush arrangement. The earth inductor may be used for measuring earth's magnetic field.

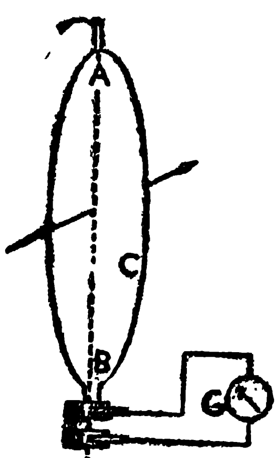


Fig. 7·2

DETERMINING EARTH'S MAGNETIC FIELD : The coil is placed in position with its plane vertical. It is now rotated uniformly. Let H be the earth's horizontal magnetic intensity. When the normal to the coil makes an angle θ with the earth's magnetic merdian,

the flux through the coil, if its effective area be $A.n$ (A is the area of each turn and n is the total number of turns), is $AnH\cos\theta$ and the induced *emf* is given by

$$e=-\frac{d}{dt}(AnH\cos\theta)$$

and the current through the coil of resistance r is

$$i=\frac{e}{r}=\frac{AnH}{r}\frac{d}{dt}(\cos\theta)\ ,\quad -ve \text{ sign being omitted}$$

Let the ends of the coil be connected to a ballistic galvanometer. If the coil is turned through 180° in a time t and q is the charge circulated through the coil in this interval, then

$$q=\int_0^t i.dt=\frac{AnH}{r}\int_0^\pi d(\cos\theta)=\frac{AnH}{r}\Big[\cos\theta\Big]_0^\pi$$

$$\text{or}\quad q=\frac{2AnH}{r},\qquad \text{hence}\quad H=\frac{qr}{2An}$$

If α is the ballistic throw $q=K\alpha$, so $H=\dfrac{Kr}{2An}\alpha$.

The vertical field V can be determined by placing the plane of the coil in a horizontal plane. If α' is the throw for such an arrangement in a similar experiment as described then $V=\dfrac{Kr}{2An}\alpha'$.

If ϕ be the angle of dip, $\tan\phi=\dfrac{V}{H}=\dfrac{\alpha'}{\alpha}$.

VII-2. AVERAGE VALUES

Average value over a half-a-cycle : The instantaneous value of an alternating *emf* is given by $E_0\ \sin pt=E_0 \sin\theta$, where p is the pulsatance. Mean value of $\sin\theta$ over a complete cycle (*i.e.* in the interval in which θ varies from 0 to 2π) is zero. So the average value of an alternating *emf* or current over a complete cycle is zero, it being positive in one and negative in the other.

Hence the mean value over half-a-cycle is generally considered as the average value. This may be obtained for an *emf* $E_0 sin\theta$ as shown below

$$\bar{E} = \frac{E_0 \int_0^{\pi} sin\theta.d\theta.}{\int_0^{\pi} d\theta} = \frac{E_0}{\pi}\Big[-cos\theta\Big]_0^{\pi} = \frac{2E_0}{\pi}$$

Similarly for an alternating current $i_0 sin\theta$, the average value is

$$\bar{i} = \frac{i_0 \int_0^{\pi} sin\theta.d\theta}{\int_0^{\pi} d\theta} = \frac{i_0}{\pi}\Big[-cos\theta\Big]_0^{\pi} = \frac{2i_0}{\pi}.$$

$$\text{So Average value} = \frac{2}{\pi} \times \text{peak value.}$$

Mean square value : An alternating current is to be measured with an instrument in which the indication of the instrument is proportional to the square of the current. If $i_0 sin\theta$ is the instantaneous value of an alternating current of peak value i_0, its mean square value over a complete cycle is obtained as

$$\overline{i^2} = \frac{\int_0^{2\pi} i_0^2 .sin^2\theta.d\theta}{\int_0^{2\pi} d\theta} = \frac{i_0^2 \int_0^{2\pi} (1-cos2\theta)d\theta}{2.2\pi} = \frac{i_0^2.2\pi}{4\pi}$$

or $\overline{i^2} = \frac{i_0^2}{2}$.

Similarly for sinusoidal *emf* $E_0 sin\theta$, the mean square value is

$$\overline{E^2} = \frac{E_0^2 \int_0^{2\pi} sin^2\theta.d\theta}{\int^{2\pi} d\theta} = \frac{E_0^2}{2}$$

Root Mean Square value or Effective value : Since the mean square value is $\frac{i_0^2}{2}$, the root mean square value of alternating current is $i_0/\sqrt{2}$. A direct or undirectional current of magnitude $i_0/\sqrt{2}$ would produce the same effect in an instrument (indicating square of the current) as an alternating current of instantaneous value $i_0 \sin\theta$. Hence the root mean square (R. M. S.) value of an alternating current is defined as the continuous or direct current of uniform intensity which would give same indication in the same measuring instrument as that produced by an alternating current. This is the *effective value* of alternating current and for a current of peak value i_0 its measure is $i_0/\sqrt{2}$. This is also called the *virtual current* as is indicated by a measuring instrument, the real value being a fluctuating one.

Hence regarding alternating *emf* or current we have the relationship

Peak value $=\sqrt{2}\times$(virtual or R.M.S. value)

The peak value of an alternating *emf* recording the virtual value 230 volts in a voltmeter is $230\times\sqrt{2}=320$ volts, a value much higher than the apparent value shown by the instrument. For a 50-cycle alternating *emf*, this value is reached 100 times in a second. Hence 230-volts A.C. supply is more dangerous for handling than a D.C. supply of same voltage.

FORM FACTOR : The ratio of effective (r.m.s) value and average value of a sinusoidal *emf* or current is called its form factor.

$$\text{Form factor}=\frac{R.M.S.\ value}{Average\ value}=\frac{(1/\sqrt{2})\times Peak\ value}{(2/\pi)\times Peak\ value}$$

$$\text{or Form factor}=\frac{\pi}{2\sqrt{2}}=1{\cdot}11$$

ILLUSTRATIVE EXAMPLES

1. Calculate the emf generated in a rectangular coil of 50 turns each of area 200 sq. cms. when rotated in a field of flux 10^{-4} weber per square centimetre at 3000 r.p.m. at an instant 0·02 second after it has passed through zero value.

Solution : $E=E_0 \sin \omega t = AnH\omega \sin \omega t$

or $E=200\times 50\times 10^{-4}\times 2\pi\times \frac{3000}{60} \times \sin [360^\circ \times 50\times \cdot 02]$

or $E=184{\cdot}5$ volts.

2. *Obtain the frequency of an alternating emf expressed as* $E=50 \sin 400\,\pi t$. *Calculate the form factor.*

Solution : Frequency $f = \frac{\omega}{2\pi} = \frac{400\pi}{2\pi} = 200$ c.p.s.

Peak value $E_0 = 50$

R.M.S. value $\frac{E_0}{\sqrt{2}} = \frac{50}{\sqrt{2}}$

Average value $\frac{2E_0}{\pi} = \frac{2\times 50}{\pi}$

Form factor $= \frac{50}{\sqrt{2}} \times \frac{\pi}{2\times 50} = \frac{\pi}{2\sqrt{2}} = 1{\cdot}11$

VII-3. A. C. THROUGH OHMIC RESISTANCE

Relation between current and emf : Let an alternating *emf* of pulsatance p and amplitude E_0 represented as $E_0 \sin pt$ be applied to a circuit of non-inductive resistance R. If i be the current at any instant, the ohmic potential drop across the resistor is iR. It is equal to the instantaneous *emf*. Hence

$$iR = E_0 \sin pt$$

$$\text{or} \quad i = \frac{E_0}{R} \sin pt = i_0 \sin pt$$

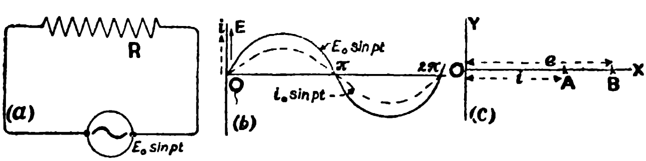


Fig. 7·3

The voltage at the ends of this resistor and the current through it are in phase at all times and they have the same frequency.

VECTOR REPRESENTATION : Alternating *emf* and current are regarded as vector quantities as these have magni-

tude and phase. So these may be represented in vector diagram. Let the current be represented in the x-axis. In the case of an *emf* driving an alternating current through ohmic resistance, the current and voltage are in phase. So voltage will also be represented along x-axis. At any instant $E=Ri$, in the diagram OA and OB represent the current (i) and the voltage drop (E) respectively. Symbolically in vector notation, $\mathbf{E}=R\mathbf{i}$.

The relation between voltage drop and current in other types of conductors, such as inductance or capacitance, will not be same and so simple as shown here. These are being treated separately.

VII-4. A.C. THROUGH INDUCTANCE

Inductive Resistance or Reactance : Let an alternating *emf* $E_0 \sin pt$ be employed to send a current through an inductance (L) having negligible ohmic resistance. When a current begins to flow, a back *emf* $L\frac{di}{dt}$ is produced which opposes the applied *emf*. In order that the current may just flow, the applied *emf* must be equal to the back *emf*, so the equation for *emf* is

$$L\frac{di}{dt} = E_0 \sin pt$$

$$\text{or} \quad di = \frac{E_0}{L} \sin pt.dt.$$

$$\text{Integrating,} \quad i = -\frac{E_0}{Lp} \cos pt + K$$

Since we are only concerned with steady state conditions, so the integration constant K is put equal to zero (See note overleaf).

$$\text{or} \quad i_0 = \frac{E_0}{Lp} \sin\left(pt - \frac{\pi}{2}\right) = i_0 \sin\left(pt - \frac{\pi}{2}\right)$$

i_0 represents E_0/Lp the amplitude of current. The quantity Lp functions as resistance (like ohmic resistance in D.C. circuits). This is called the ***inductive resistance*** or ***reactance***. The potential drop at the ends of the inductance is Lpi, wher

a current i having a frequency $p/2\pi$ flows through it. The

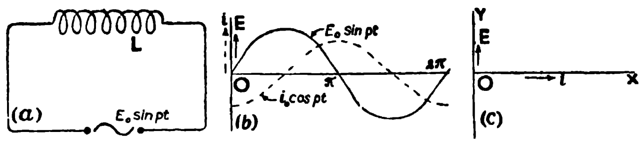


Fig. 7·4

emf, $(E_0 sin\, pt)$, leads the current $i_0 sin\left(pt-\frac{\pi}{2}\right)$ by $\frac{\pi}{2}$. It means that the current reaches its peak value one quarter of of a period later. The *emf* and current are said to be in *quadrature*. The relation between the peak values of the two are related as $E_0 = Lpi_0$ and the virtual values as $E = Lpi$. The magnitude of inductive reactance depends upon the frequency of the applied *emf*. It is denoted as $X_L = Lp$. We may consider that for unidirectional *emf*, since $p=0$, $X_L = 0$, that is an inductance offers no resistance to direct current.

Note : The integrating constant in the above treatment has been taken to be zero but the reason may not be very convincing. This may be avoided if we proceed in the reverse way. Let $i_0 cos\, pt$ be the current in a circuit containing an inductance L and no ohmic resistance. The *emf* induced is $e = -L\frac{di}{dt} = Lpi_0 sin\, pt$. In order that the current may flow the applied *emf* should be always equal and opposite to the induced *emf*, hence the applied *emf* at any instant is

$$E = -Lpi_0 sinpt = Lpi_0 cos\left(pt+\frac{\pi}{2}\right)$$

$$\text{or} \quad E = E_0 \; cos\left(pt+\frac{\pi}{2}\right)$$

The applied voltage leads the current by $\frac{\pi}{2}$.

VECTOR DIAGRAM : The alternating *emf* or current has magnitude and phase. As such they behave as vectors. The convention accepted is that an angle of lead is measured in counter-clockwise direction in a vector diagram. Thus if the current is represented in x-axis, the *emf* developed at the

ends of inductance should be represented in the y-axis, since in this case the current lags the *emf* by an angle $\pi/2$.

SYMBOLIC REPRESENTATION : Since mathematically a vector multiplied by -1 indicates a rotation through an angle π, rotation through an angle $\pi/2$ is considered as multiplication by $\sqrt{-1}$ represented by j. (This is quite logical since $\sqrt{-1}\times\sqrt{-1}=-1$ and rotation twice through $\pi/2$ amounts to a rotation through π). The relation between two vectors E and i as occuring in a flow through an inductance in A.C. circuits is expressed in mathematical symbols in vector notation as $\mathbf{E}=jLp\mathbf{i}$, as distinguished from a similar relation ($E=Ri$) in an ohmic resistance.

VII-4. A.C. THROUGH CAPACITANCE

Mechanism of conduction : A capacitor included in a circuit in which an alternating *emf* has been applied does not cause a blocking of the flow of current, though the conducting plates of the condenser is intervened by a non-conducting dielectric medium.

The manner in which the alternating current in the external circuit is continued even due to the presence of a capacitor in the circuit may be realised by considering a hydraulic analogy.

A tube forming a closed ring having a bulb separated in two halves by an elastic membrane is fitted with a pump and the tube is filled with water. A pressure applied to the piston causes water in the tube to move. This rush of water in the expansion chamber causes pressure on the membrane which again drives the water on the opposite side to flow through the ring. Thus there is water-flow through the tube on both sides of the membrane though water does not pass through the partition. On relasing the pressure on the piston the membrane returns to its undisturbed position

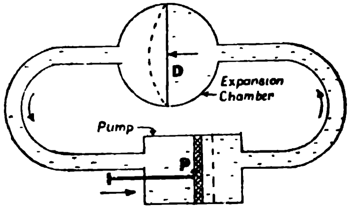


Fig. 7·5

causing displacement of water in both halves now in the opposite direction. A to-and-fro motion of the piston causes similar flow of water on either side of the membrane though there is no transference of water from one side of the membrane to the other. Flow of charge in a capacitor circuit is similar to this in character.

When an *emf* is applied to the plates of a capacitor there is a flow of charge through the conducting wire and the plates are first charged in a pattern. When the *emf* is reversed the charge from the plates flows through the wire in opposite direction and the plates are charged in the reversed way. The result is to-and-fro motion of electrons externally between the plates and the source through the wire. Thus there is an alternating current on either side of the capacitor though there is no transfer of charge through the dielectric.

Capacitative Reactance : Let C be the capacitance of a of a condenser put across a source of alternating *emf* $E_0 sin\ pt$. If Q be the charge acquired by the condenser at any instant, the potential difference between the plates is Q/C, hence

$$\frac{Q}{C} = E_0 sinpt$$

$$\text{or} \quad Q = E_0 C\ sinpt$$

$$\text{or} \quad i = \frac{dQ}{dt} = E_0 Cp\ cos\ pt = \frac{E_0}{1/Cp}\ sin\left(pt + \frac{\pi}{2}\right)$$

$$\text{or} \quad i = i_0\ sin\left(pt + \frac{\pi}{2}\right)$$

i_0 is the amplitude of the current produced, represented as $\frac{E_c}{1/Cp}$. The current leads the voltage at the plates of the

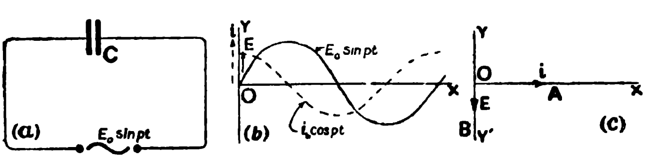


Fig. 76

capacitor by $\pi/2$ at any instant and the virtual values are

realised as $E = i/Cp$. The quantity $1/Cp$ behaves as resistance occuring in Ohm's law. Writing the *capacitative resistance or reactance* as $X_c = 1/Cp$, we have $E = iX_c$.

The capacitative reactance depends upon p, that is upon the frequency of the applied *emf*. Hence a condenser or capacitor of definite capacitance offers different resistances to alternating *emfs* of different frequencies. For unidirectional *emf*, $p = 0$, hence $X_c = \infty$. So condensers are found to offer infinite resistance to direct current. When the frequency is such that a condenser offers a negligible resistance it is a '*by-pass*' condenser and when for a high value of frequency the reactance is high the same condenser acts as a '*blocking*' condenser. The potential drop at the plates is $E = i/Cp$.

VECTOR DIAGRAM : To represent the relationship between the current and the voltage in a vector diagram (Fig. 7·6c) if OA is drawn along x-axis to represent the current, OB drawn along the negative direction of y-axis on the same scale would represent the voltage at the condenser at the same instant, since an angle of lag in phase is measured in clockwise direction.

IN SYMBOLIC NOTATION : Reactance drop in a capacitor is given by i/Cp lagging in phase by $\frac{\pi}{2}$ with respect to current and hence in symbolic notation it is written as

$$\frac{i}{jCp} = -\frac{ij}{Cp}$$

In vector notation, $\mathbf{E} = -\dfrac{j\,\mathbf{i}}{Cp}$

The capacitative reactance is considered negative in relation to inductive reactance.

VII-5. MIXED CIRCUITS

Case I—Resistance and Inductance in series : Let E_0 *sin pt* be the applied *emf* in a circuit containing a resistance (R) in series with an inductance (L). Let i be the instantaneous

current. The ohmic drop of potential at the ends of R is $V_R = Ri$ and the reactance drop at the ends of the inductance is $V_L = Lpi$. The current in relation to the applied *emf* may be obtained in different ways.

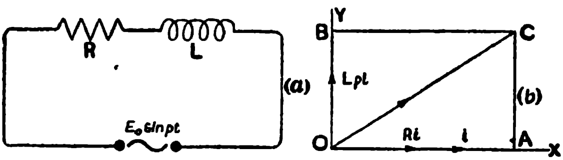


Fig. 7·7

(i) USING VECTOR DIAGRAM : As shown in the previous section (§ VII-3) the potential drop in the resistor is in phase with the current and that in the inductance (§ VII-4) is $\pi/2$ ahead of the current. These two of magnitudes Ri and Lpi respectively represented by OA and OB are drawn to scale along the x-axis and the y-axis (Fig. 7·7b). Hence the resultant potential drop in the circuit (which is equal to the applied *emf*) is obtained as the vector sum of the two.

Hence as shown in the diagram

$$\mathbf{OC} = \mathbf{OA} + \mathbf{OB}$$

$$\mathbf{E} = \mathbf{i}\sqrt{R^2 + L^2p^2}$$

$$\text{or} \quad \mathbf{i} = \frac{\mathbf{E}}{\sqrt{R^2 + L^2p^2}}$$

This relation applies as well to magnitudes of peak and virtual (*r. m. s.*) values in the forms

$$i_0 = \frac{E_0}{\sqrt{R^2 + L^2p^2}} \quad \text{and} \quad i_{r.m.s.} = \frac{E_{r.m.s.}}{\sqrt{R^2 + L^2p^2}}$$

The current lags behind the applied *emf* by an angle ϕ, given by

$$\phi = \tan^{-1}\left(\frac{AC}{AO}\right) = \tan^{-1}\frac{Lp}{R}.$$

Hence the magnitudes of current and *emf* at any instant is related as

$$i = \frac{E_0 \sin (pt-\phi)}{\sqrt{R^2+L^2p^2}}$$

The quantity $\sqrt{R^2+L^2p^2}$ is called the *impedance* of the circuit. This functions as ohmic resistance in *D. C.* circuits.

(ii) USING SYMBOLIC NOTATION : The voltage drop **E** at the ends of the combination is obtained by considering the ohmic and reactance drops of potential. So

$$\mathbf{E} = R\mathbf{i}+jLp\mathbf{i}$$

$$\text{or} \quad \mathbf{E} = \mathbf{i}(R+jLp)$$

$$\text{or} \quad \mathbf{i} = \frac{\mathbf{E}}{R+jLp}$$

$R+jLp$ is the vector operator of the *impedance* Z. The magnitude of impedance is $\sqrt{R^2+L^2p^2}$. Thus the peak and virtual values of *emf* and current are connected by the formulae

$$i_0 = \frac{E_0}{\sqrt{R^2+L^2p^2}} \quad \text{and} \quad i_{r.m.s.} = \frac{E_{r.m.s.}}{\sqrt{R^2+L^2p^2}}$$

The phase difference between the voltage and the current is given by $\phi = \tan^{-1}\left(\frac{Lp}{R}\right)$. Hence the relation between the amplified *emf* and the current in the circuit is given by

$$i = \frac{E_0 \sin (pt-\phi)}{\sqrt{R^2+L^2p^2}}$$

(iii) USING DIFFERENTIAL EQUATION : Let $E_0 \sin pt$ be the voltage applied at the ends of a resistance (R) and an inductance (L) in series. Let i be the instantaneous current. The ohmic drop of potential at the ends of R is Ri and the reactance drop at the ends of L is $L\frac{di}{dt}$. Hence in order that the current may just flow, the condition to be satisfied is given by

$$Ri = E_0 \sin pt - L\frac{di}{dt}$$

$$\text{or} \quad L\frac{di}{dt}+Ri = E_0 \sin pt$$

$$\text{or} \qquad \frac{di}{dt}+\frac{R}{L}i = \frac{E_0}{L}\sin pt$$

The value of i the current at any instant is to be obtained by solving this differential equation. Multiplying by the intregrating factor $e^{\frac{R}{L}t}$, we get

$$e^{\frac{R}{L}t}\cdot\frac{di}{dt}+e^{\frac{R}{L}t}\cdot\frac{R}{L}i = e^{\frac{Rt}{L}}\frac{E_0}{L}\sin pt$$

$$\text{or} \qquad \frac{d}{dt}\left(e^{\frac{R}{L}t}i\right) = \frac{E_0}{L}e^{\frac{R}{L}t}\sin pt$$

$$\text{Integrating,} \quad e^{\frac{Rt}{L}}i=\frac{E_0}{L}\frac{e^{\frac{R}{L}t}\sin\left(pt-\tan^{-1}\frac{Lp}{R}\right)}{\sqrt{\frac{R^2}{L^2}+p^2}}+K$$

$$\text{or} \quad i=\frac{E_0}{\sqrt{R^2+L^2p^2}}\sin\left(pt-\tan^{-1}\frac{Lp}{R}\right)+Ke^{-\frac{R}{L}t}$$

when $t\to\infty$, $e^{-\frac{R}{L}t}$ vanishes, hence in the steady state current at any instant is given by

$$i=\frac{E_0}{Z}\sin(pt-\phi)=i_0\sin(pt-\phi)$$

where $Z=\sqrt{R^2+L^2p^2}$ and $\tan\phi=\frac{Lp}{R}$. The amplitude of the current $i_0=\frac{E_0}{\sqrt{R^2+L^2p^2}}$. The quantity $\sqrt{R^2+L^2p^2}$ is the *impedance* of the circuit and it is the vector sum of the resistance and reactance. The magnitudes of virtual and the peak values are obtained from the equation $i=E/Z$ and $i_0=E_0/Z$. The current lags behind the applied *emf* by an angle $\tan^{-1}(Lp/R)$.

Case 2—Resistance and Capacitance in series : Let R and X_0 be the ohmic and capacitative resistances. In a circuit containing a resistor R and a capacitance C the potential drop at the ends of the resistor is $V_R=iR$ in phase with the current i and the potential drop at the ends of the capacitance is

$V_X = iX_c = i/Cp$ and it is in phase $\pi/2$ behind the current. The relation between the current in the circuit at any instant and the applied *emf* $E_0 \sin pt$ may be obtained by different methods.

(i) USING A VECTOR DIAGRAM : As shown in the vector diagram (Fig. 7·8b) **OA** represents V_R along x-axis and **OB** represents V_X along y-axis in the negative direction. The resultant potential drop is the vector sum of V_R and V_X, *i.e.*

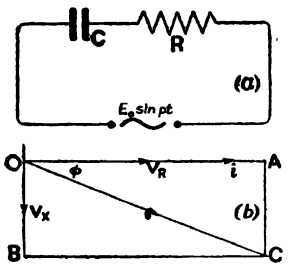


Fig. 7·8

$$E = OC = OA + OB$$

$$= i\sqrt{R^2 + \left(\frac{1}{Cp}\right)^2} = iZ$$

The phase lag of the current behind the applied *emf* is given by

$$\phi = \tan^{-1}\left(\frac{AC}{AO}\right) = \tan^{-1}\left(\frac{-X_c}{R}\right) = -\tan^{-1}\left(\frac{1}{RCp}\right)$$

ϕ is negative and so the current leads the *emf*. The current at any instant is given by

$$i = \frac{E_0 \sin (pt + \phi)}{\sqrt{R^2 + \left(\frac{1}{Cp}\right)^2}} = \frac{E}{Z}, \quad Z \text{ is the impedance}$$

The virtual and peak values of current and *emf* are related as, $i = E/Z$ and $i_0 = E_0/Z$.

(ii) USING SYMBOLIC NOTATION : The voltage drop at the ends of the combination is obtained by considering the ohmic and capacitative drops, hence

$$E = iR - \frac{ij}{Cp} = i\left(R - \frac{j}{Cp}\right)$$

$$\text{or} \quad i = \frac{E}{R - j/Cp}$$

$R - j/Cp$ is the vector operator of the impedance Z, the magnitude of which is $\sqrt{R^2 + (\frac{1}{Cp})^2}$. So the magnitude of the current is

$$i = \frac{E}{\sqrt{R^2 + (1/Cp)^2}}$$

The phase difference between the applied *emf* and the current is $\phi = -tan^{-1}\left(\frac{1}{RCp}\right)$ and the current leads the *emf*.

(iii) USING DIFFERENTIAL EQUATION: An *emf* $E_0 sinpt$ is applied at the plates of a condenser of capacitance C through an ohmic resistance R. If i is the instantaneous current when Q is the charge on C, then the potential drops in R and C are together equal to the applied *emf*, hence

$$Ri = E_0 sin\,pt - \frac{Q}{C}$$

or $$Ri + \frac{Q}{C} = E_0 sin\ pt$$

Differentiating, $R\frac{di}{dt} + \frac{i}{C} = E_0 p cos\ pt$, since $\frac{dQ}{dt} = i$

or $$\frac{dt}{dt} + \frac{i}{CR} = \frac{E_0 p}{R}\ cos\ pt$$

Multiplying by the integrating factor $e^{\frac{t}{CR}}$,

$$\frac{di}{di}\ e^{\frac{t}{CR}} + \frac{i}{CR}\ e^{\frac{t}{CR}} = \frac{E_0 p}{R} e^{\frac{t}{CR}}\ cos\ pt$$

Integrating, $$ie^{\frac{t}{CR}} = \frac{E_0 p e^{\frac{t}{CR}}\ cos\,[pt - tan^{-1}(CpR)]}{R\sqrt{p^2 + (1/CR)^2}} + K$$

$$i = \frac{E_0\ sin\,[pt + tan^{-1}(\frac{1}{CpR})]}{\sqrt{R^2 + (\frac{1}{Cp})^2}} + Ke^{-\frac{t}{CR}}$$

or $$i = \frac{E_0 sin[pt + tan^{-1}(1/CpR)]}{\sqrt{R^2 + (1/Cp)^2}}\ \text{when } t \to \infty$$

or $$i = \frac{E_0 sin(pt + \phi)}{Z},$$

where $tan\ \phi = \frac{1}{CpR}$ and $Z^2 = R^2 + (1/Cp)^2$. The current leads the *emf* by an angle $tan^{-1}(1/CpR)$. The amplitude and the virtual values are given by

$$i_0 = \frac{E_0}{Z}\ \text{and}\ i_{r.m.s.} = \frac{E_{r.m.s.}}{Z}$$

Case 3—Inductance and Capacitance in series : Let $E_0 \sin pt$ be the applied *emf* in a circuit containing an inductance (L) and a capacitance (C) in series. If i be the current at any instant, the reactance drops respectively are Lpi and $-i/Cp$, both differing in phase by $\frac{\pi}{2}$ with respect to the current, the former leading and the other lagging. Let OA and OB represent these two in the vector diagram,

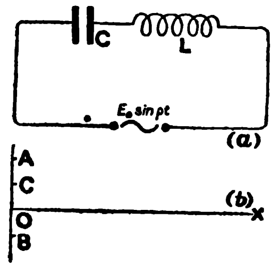


Fig. 7·9

$OA = Lpi$ and $OB = i/Cp$. The resultant potential drop is $(OA - OB) = OC$, hence

$$i = \frac{E}{Lp - \frac{1}{Cp}}.$$

The current lags behind the voltage or leads it by $\frac{\pi}{2}$ according as $Lp - \frac{1}{Cp}$ is positive or negative.

Case 4—General (L-R-C) series circuit : A resistance R is in series with an inductance L and capacitance C and an alternating *emf* $E_0 \sin pt$ is applied to it. To obtain the instantaneous current, we may proceed in different ways as shown below.

(i) USING A VECTOR DIAGRAM : Let the current be represented in x-axis. The ohmic drop of potential Ri is in phase with current i and is represented by OA in the vector diagram in magnitude and direction. The inductive reactance drop Lpi in the inductance leads the current by $\pi/2$ and hence it is represented along y-axis by OB. The capacitative reactance drop i/Cp in the capacitor lags the current by $\pi/2$ and is represented along y-axis in the negative sense by OC. The resultant

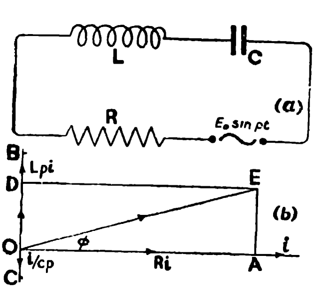


Fig. 7·10

potential drop in the inductive and capacitative circuits is obtained as difference of OB and OC, shown in the vector diagram as OD. The effective potential drop in the L-R-C circuit is obtained as the vector sum of **OA** and **OD** represented by **OE** in the vector diagram. The magnitude of OE is $\sqrt{OA^2+OD^2}$, hence in magnitude

$$E = i\sqrt{R^2+\left(Lp-\frac{1}{Cp}\right)^2}$$

$$\text{or}\quad i = \frac{E}{\sqrt{R^2+\left(Lp-\frac{1}{Cp}\right)^2}}$$

$$\text{Impedance}\quad Z = \sqrt{R^2+\left(Lp-\frac{1}{Cp}\right)^2}$$

The phase difference between the applied *emf* and the current being given by

$$\phi = \tan^{-1}\left(\frac{AE}{AO}\right) = \tan^{-1}\left[\frac{Lp-\frac{1}{Cp}}{R}\right]$$

The instantaneous value of the current is

$$i = \frac{E_0 \sin(pt-\phi)}{\sqrt{R^2+(Lp-\frac{1}{Cp})^2}}$$

The amplitude of current $i_0 = \dfrac{E_0}{Z}$

The virtual current $i_{r.m.s.} = \dfrac{E_{r.m.s.}}{Z}$

(ii) USING SYMBOLIC NOTATION: The ohmic potential drop is $R\mathbf{i}$ and the reactance drops in the inductance and capacitance are respectively $jLp\mathbf{i}$ and $\dfrac{-j}{Cp}\mathbf{i}$. Hence the instantaneous *emf* **E** is given by

$$\mathbf{E} = R\mathbf{i}+jLp\mathbf{i} - \frac{j}{Cp}\mathbf{i} = \left(R+jLp-\frac{j}{Cp}\right)\mathbf{i}$$

$$\mathbf{i} = \frac{\mathbf{E}}{R+j(Lp-\frac{1}{Cp})}$$

$R+j(Lp-\frac{1}{Cp})$ is the vector operator for the impedance of magnitude obtained as $Z=\sqrt{R^2+(Lp-\frac{1}{Cp})^2}$. The magnitude

of $i=E/Z$. This expression applies to peak and virtual values. The phase difference between the applied voltage and the current produced is $\phi=tan^{-1}\left[\frac{Lp-\frac{1}{Cp}}{R}\right]$. Hence the magnitude of instantaneous current is

$$i=\frac{E_0 sin(pt-\phi)}{\sqrt{R^2+(X_L-X_0)^2}}, \quad \text{where } X_L=Lp,\ X_C=\frac{1}{Cp}$$

(iii) USING DIFFERENTIAL EQUATION : The equation of the potential drops in a circuit containing a resistance R in series with an inductance L and a capacitance C in which an applied *emf* $E_0 sin\ pt$ produces a current i, may be written as

$$L\frac{di}{dt}+Ri+\frac{Q}{C}=E_0 sin\ pt$$

The value of the current is obtained by solving the equation. For this purpose in the equation, E_0e^{jpt} is written for $E_0 sin\ pt$, which is the imaginary part of E_0e^{jpt}.

$$L\frac{di}{dt}+Ri+\frac{Q}{C}=E_0e^{jpt}$$

$$\text{or} \quad L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C}=jpE_0e^{jpt}$$

Let $i=Ae^{jpt}$ be a solution.

Then $\frac{di}{dt}=jpAe^{jpt}$, $\frac{d^2i}{dt^2}=-p^2Ae^{jpt}$. Putting these values in the equation we get,

$$-p^2LA+jpRA+\frac{A}{C}=jpE_0$$

$$\text{or} \quad jpLA+RA+\frac{A}{jCp}=E_0$$

$$\text{or} \quad A=\frac{E_0}{R+j\left(Lp-\frac{1}{Cp}\right)}=\frac{E_0\left[R-j\left(Lp-\frac{1}{Cp}\right)\right]}{R^2+\left(Lp-\frac{1}{Cp}\right)^2}$$

$$\text{or} \quad A=\frac{E_0}{\sqrt{R^2+\left(Lp-\frac{1}{Cp}\right)^2}}\left[\frac{R}{\sqrt{R^2+\left(Lp-\frac{1}{Cp}\right)^2}}-\frac{j\left(Lp-\frac{1}{Cp}\right)}{\sqrt{R^2+\left(Lp-\frac{1}{Cp}\right)^2}}\right]$$

Writing $Z=\sqrt{R^2+(Lp-\frac{1}{Cp})^2}$, $\frac{R}{Z}=\cos\phi$ and

$$\frac{Lp-\frac{1}{Cp}}{Z}=\sin\phi\ ,\quad \tan\phi=\frac{Lp-\frac{1}{Cp}}{R}$$

We have $A=\frac{E_0}{Z}\left[\cos\phi-j\sin\phi\right]=\frac{E_0}{Z}e^{-j\phi}$

Hence $i=Ae^{jpt}=\frac{E_0}{Z}e^{j(pt-\phi)}$

Considering the imaginary part only

$$i=\frac{E_0}{Z}\sin(pt-\phi)$$

Impedance Z is the vector sum of resistance R and reactance (X_L and X_C) *i.e.* $Z=\sqrt{R^2+(X_L-X_C)^2}$, where $X_L=Lp$ and $X_C=1/Cp$. *Emf* leads the current by ϕ where $\tan\phi=\frac{X_L-X_C}{R}$. ϕ is negative if $X_L<X_C$ and in that case current leads the *emf.* The peak and the virtual values are obtained as $i_0=E_0/Z$ and $i=E/Z$ respectively.

Particular cases : (i) When $L=0$, $C=\infty$ then $\phi=0$ and $\tan\phi=0$, $i=\frac{E_0}{R}\cos pt$ [as shown in § VII-3].

(2) When $C=\infty$, $\tan\phi=\frac{Lp}{R}$, $i=\frac{E_0\sin(pt-\phi)}{\sqrt{R^2+L^2p^2}}$

[as shown in § VII-5(*i*)]

(3) When $L=0$, $\tan\phi=\frac{1}{CpR}$, $i=\frac{E_0\sin(pt-\phi)}{\sqrt{R^2+(1/Cp)^2}}$.

[as shown in § VII-5(2)]

(4) When $Lp=\frac{1}{Cp}$, $\phi=0$, $i=\frac{E_0}{R}\cos pt$

[as shown in § VII-7]

ILLUSTRATIVE EXAMPLES

1. Calculate the current flowing through an inductance of 0·4 henry and of negligible resistance when an emf of 200 volts at 50 c.p.s. is applied at its ends.

Solution : $X_L = Lp = 2\pi fL = 2\pi \times 50 \times 0{\cdot}4 = 125{\cdot}6$ ohms

$$\text{So } i = \frac{200}{125{\cdot}6} = 1{\cdot}592 \text{ amps.}$$

2. A coil of resistance 50 ohms and inductance 0·05 henry is joined with the terminals of a source of alternating emf of 230 volts at 50 c.p s. Calculate the current and the phase difference between the emf and current.

Solution : $R = 50$ ohms, $X_L = (0{\cdot}05 \times 2\pi \times 50) = 15{\cdot}7$ ohms

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{50^2 + (15{\cdot}7)^2} = 55{\cdot}2 \text{ ohms}$$

$$i = \frac{230}{55{\cdot}2} = 4{\cdot}16 \text{ amp.}$$

$$\text{Phase } \phi = \tan^{-1}\left(\frac{15{\cdot}7}{50}\right) = \tan^{-1}(0{\cdot}314)$$

$$\text{or } \phi = 17^\circ{\cdot}4 \ (\textit{emf} \text{ leading})$$

3. A 200-volt-50 cycle A.C. supply is connected to a circuit containing a resistance of 20 ohms in series with a 100μF capacitor. Calculate the current and phase.

Solution : $R = 20$ ohms, $X_c = \dfrac{10^6}{2\pi \times 50 \times 100} = 31{\cdot}8$ ohms.

$$Z = \sqrt{R^2 + X_c^2} = \sqrt{20^2 + (31{\cdot}8)^2} = 37{\cdot}5 \text{ ohms.}$$

$$i = \frac{200}{37{\cdot}5} = 5{\cdot}6 \text{ amps.}$$

$$\phi = -\tan^{-1}\left(-\frac{31{\cdot}8}{20}\right) = 58^\circ \text{ (current leading)}$$

4. Obtain the impedance at 50 c.p.s. of the circuit which contains a resistance of 20 ohms in series with an inductance of 0·1 henry and a capacitance of 200μF.

Solution : $R = 20$ ohms, $X_L = 2\pi \times 50 \times 0{\cdot}1 = 31{\cdot}4$ ohms

$$X_c = \frac{10^6}{2\pi \times 50 \times 200} = 15{\cdot}9 \text{ ohms}$$

$$Z = \sqrt{R^2 + (X_L - X_c)^2} = \sqrt{20^2 + (15{\cdot}5)^2} = 25{\cdot}1 \text{ ohms.}$$

VII-6. POWER CONSUMPTION

Power in resistor circuit : In a circuit having ohmic resistance only an *emf* E_0 *sinpt* causes a current i_0 *sinpt* in

phase with the *emf*. The power consumed in the circuit is given by

$$P = Ei = E_0 sinpt.\, i_0 sinpt = E_0 i_0\, sin^2 pt$$

$$\text{Mean Power } \overline{P} = E_0 i_0 . \frac{1}{2} = \frac{E_0}{\sqrt{2}} \cdot \frac{i_0}{\sqrt{2}}$$

So $\overline{P}$ = virtual *emf* × virtual current.

So average power consumed in a resistor carrying alternating current taken over a complete cycle is obtained as the product of the ammeter and voltmeter readings. The power appears in the form of heat production.

Power in inductive circuit : An alternating *emf* $E_0 sinpt$ drives a current $i_0 cospt$ through an inductance. The power consumed in the inductance is given by

$$P = E \cdot i = E_0 sin\, pt.\, i_0 cospt = E_0 i_0\, sinpt\, cospt$$

$$\text{or} \quad P = \tfrac{1}{2}\, E_0 i_0\, sin\, 2pt$$

Average value of P over a complete cycle is therefore zero. Hence the flow of alternating current through an inductance causes no power dissipation inside it.

WATTLESS CURRENT : At alternating current flowing through an inductance causes the creation of a magnetic field. Work has to be done for this and the circuit supplies positive power. This goes on till the current reaches it maximum value. The power absorbed so far is $\frac{1}{2}E_0 i_0$. After one-quarter period, the current and the magnetic field diminishes. This involves a supply of power by the coil to the circuit *i.e.* the power is negative. Thus when the magnetic field is zero again, the total work is zero. Consequently the average power is zero. During the first and third quarters *i.e.* during growth of current energy is supplied by the source to the coil (for establishing a flux) but in the second and fourth quarters *i.e.* during decay of current an equal amount af energy is returned by the coil (during withdrawal of flux). Thus the total energy supplied in a cycle is zero. Current flows to-and-fro in the circuit but on the average no work is obtained in the process. Such a current is known as wattless current. It may be noted that if there be a secondary linked with the coil,

the condition becomes different and the current through the coil no longer remains wattless.

CHOKE COIL : A coil having inductance offers resistance to the flow of alternating current through it. So an inductance may be used to control the alternating current. An inductance used as such is called a *choke.* Ohmic resistors used to reduce the current in a circuit causes dissipation of energy in the form of heat in the coil. Such useless production of energy can be avoided in an *A.C.* circuit by using an inductance in series with the source and the appliance using electric current. It would control the current as may be necessary but would consume no energy within itself excepting hysteresis loss if the inductance has an iron core. Since the inductive reactance is given by the product of inductance and frequency ($X_L = Lp$) inductive resistances used for control of current are distinguished as *high frequency* and *low frequency* chokes.

Power in L-R-C circuit : If a sinusoidal *emf* $E_0 sin\, pt$ be applied to a circuit having impedance, the current is obtained in the form $i_0 sin\,(pt-\phi)$ where $\phi = tan^{-1}\left(\frac{X_L - X_C}{R}\right)$. A phase

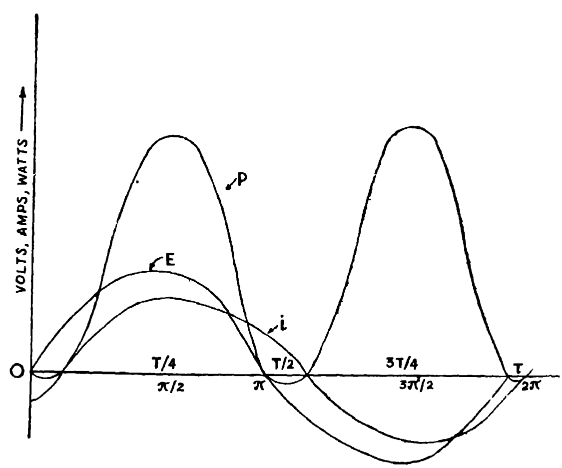


Fig 7·11

difference exists between the applied *emf* and the current produced. In such a circuit power is given by

$$P = E_0 \sin pt \; i_0 \sin(pt - \phi)$$

or $P = E_0 i_0 \, [\, \sin^2 pt \cos\phi - \sin pt \cos pt \cos\phi \,]$

or $P = E_0 i_0 \, [\, \sin^2 pt \cos\phi - \frac{1}{2} \sin 2pt \cos\phi \,]$

The nature of variation of *emf*, current and power in *L-R-C* circuit with time is shown in the diagram (Fig. 7·11).

Average power over a complete cycle is obtained as

$$\bar{P} = E_0 i_0 \, [\, \tfrac{1}{2} \cos\phi - 0 \,]$$

or $\bar{P} = \frac{E_0}{\sqrt{2}} \cdot \frac{i_0}{\sqrt{2}} \cos\phi$

or $\bar{P} = \textit{virtual volts} \times \textit{virtual amperes} \times \cos\phi$

This gives the ***true power*** absorbed. The magnitude of power obtained by taking the product of ammeter and voltmeter readings (giving the virtual values) is the ***apparent power.*** The ratio of these two is called the power factor.

$$\text{Power factor} = \frac{\textit{True power}}{\textit{Apparent power}} = \cos\phi = \frac{R}{Z}$$

WATTLESS COMPONENT : Let the amplitudes of current i_0 and voltage E_0 be represented in a vector diagram by straight lines OA and OB respectively inclined mutually at an angle ϕ, the phase difference. Let the current represented by OA be resolved along OB and at right angles to it. Let OD and OC be the components respectively of magnitudes $i_0 \cos\phi$ and $i_0 \sin\phi$.

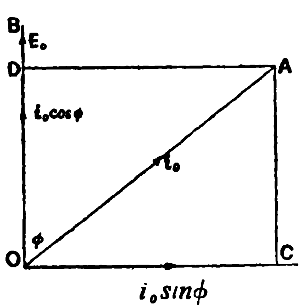


Fig. 7·12

To obtain the power due to the two components we are to consider the scalar product of the *emf* and the relevant component. The scalar product of E_0 and $i_0 \sin\phi$ is zero, since they are at right angles. So this component contributes nothing to the power consumption. Hence it is called ***wattless*** or ***reactive*** component. The other component $i_0 \cos\phi$ acting in the same line as E_0 consumes a power $E_0 i_0 \cos\phi$ and this is the ***active*** or ***wattful*** component.

POWER FACTOR IN DIFFERENT CIRCUITS : Power factor will be different in circuits having different components. In any circuit power factor $cos\phi=\frac{R}{Z}$ and hence $cos\phi$ depends upon Z, the impedance and R, the resistance.

In *inductive resistor coils* the impedance $Z=\sqrt{L^2p^2+R^2}$.

$$\text{So} \quad cos\phi=\frac{R}{Z}=\frac{R}{\sqrt{L^2p^2+R^2}}.$$

In *capacitative circuits* the phase-lead of the current over the *emf* is $\pi/2$ and as such the power should be zero. But imperfect capacitance always brings in some loss of energy. The imperfection is due to bad insulation, heating effects in the leads and plates and dielectric absorption. In such a case the energy loss is considered by introducing an equivalent resistance. If the total resistance (considering the external resistance and the equivalent resistance) be R and capacitance be C, then

$$Z=\sqrt{R^2+(\tfrac{1}{Cp})^2}, \text{ hence } cos\phi=\frac{R}{Z}=\frac{R}{\sqrt{R^2+(\frac{1}{Cp})^2}}$$

If R^2 be negligible in presence of $(\frac{1}{Cp})^2$, then $cos\phi=CPR$.

If *L-R-C series circuit*, the impedance is given by

$$Z=\sqrt{R^2+(Lp-\tfrac{1}{Cp})^2}, \text{ hence}$$

$$cos\phi=\frac{R}{Z}=\frac{R}{\sqrt{R^2+(X_L-X_C)^2}}$$

ILLUSTRATIVE EXAMPLES

1. An arc lamp works at 50 volts and consumes 10 amperes. If it is run by 200volt 50 cycle A.C. supply, calculate the value of the choke to be inserted in the circuit.

Solution : Resistance of the arc$=\frac{50}{10}=5$ ohms.

Using $i=\frac{E}{\sqrt{L^2p^2+R^2}}$, $10=\frac{200}{\sqrt{L^2p^2+R^2}}$,

Hence $L^2p^2+R^2=400$

or $L^2p^2=400-R^2=400-25=375$

or $Lp=19{\cdot}3$ ohm

So $L=\dfrac{19{\cdot}3}{2\pi\times 50}=\dfrac{19{\cdot}3}{314}=0{\cdot}061$ henry.

2. *An alternating emf of 200 volts (r.m.s. value) at 50 c.p.s. is applied to a circuit containing a resistance of 40 ohms in series with an inductance of 0·1 henry. Calculate the power consumption and the potential drops along the resistor and the inductance.*

Solution : Impedance $Z=\sqrt{R^2+L^2p^2}$

$=\sqrt{40^2+(2\times 50\times\pi\times 0{\cdot}1)^2}$

or $Z=50{\cdot}86$ ohms.

$$i=\frac{E}{Z}=\frac{200}{50{\cdot}86}=3{\cdot}93 \text{ amps.}$$

$V_R=Ri=40\times 3{\cdot}93=157{\cdot}2$ volts

$V_L=Lpi=31{\cdot}4\times 3{\cdot}93=123{\cdot}4$ volts

$$cos\phi=\frac{R}{Z}=\frac{40}{50{\cdot}86}=0{\cdot}78$$

$P=Eicos\phi=200\times 3{\cdot}93\times 0{\cdot}78=613{\cdot}08$ watts.

3. *A capacitance of 500 μF is in series with a resistance of 20 ohms. Calculate the current and power factor when a 200-volt-50 cycle A.C. supply is applied.*

Solution : $P=2\pi f=2\pi\times 50=314$

Reactance $X_c=\dfrac{1}{Cp}=\dfrac{10^6}{500\times 314}=6{\cdot}36$ ohms.

Impedance $Z=\sqrt{20^2+(6{\cdot}36)^2}=25{\cdot}37$ ohms,

Current $i=\dfrac{E}{Z}=\dfrac{200}{25{\cdot}37}=7{\cdot}88$ amps

Power factor $cos\phi=\dfrac{R}{Z}=\dfrac{20}{25{\cdot}37}=0{\cdot}788$

4. *Calculate the reactance of the coil which takes 2 amperes with an applied emf of 10 volts at a power factor of 0·8.*

Solution : $i=\dfrac{E}{Z}$, so $Z=\dfrac{E}{i}=\dfrac{10}{2}=5$ ohms.

Again $\quad cos\ \phi=\frac{R}{Z}$, so $R=Z\ cos\ \phi=5\times0{\cdot}8=4$ ohms

Reactance $\quad X=\sqrt{Z^2-R^2}=\sqrt{5^2-4^2}=3$ ohm.

5. Two mills have a total load of 400 ampers at a power factor of 0·6 (lagging) when they are connected with the same A.C. supply. If one of them consumes 250 ampers at power factor of 0·8 (lagging), calculate the load and the power factor of the other.

Solution : Total load $\quad i=400$ amps, $cos\ \phi=0{\cdot}6$

Active component $i\ cos\ \phi=400\times0{\cdot}6=240$ amp.

Wattless component $i\ sin\ \phi=400\sqrt{1-cos^2\phi}=400\times0{\cdot}8$

$=320$ amp.

For one mill $i_1=250$ amp., $cos\ \phi_1=0{\cdot}8$

Active component $i_1\ cos\ \phi_1=250\times0{\cdot}8=200$ amp.

Wattless component $i_1\ sin\ \phi_1=250\times0{\cdot}6=150$ amp.

For the other mill, $i_2\ cos\phi_2=i\ cos\phi-i_1\ cos\phi_1=240-200$

$=40$ amp

and $i_2\ sin\phi_2=i\ sin\phi-i_1\ sin\phi_1=320-150$

$=170$ amp

$$i_2=\sqrt{i_2{}^2\ cos^2\phi_2+i_2{}^2\ sin^2\phi_2}=\sqrt{40^2+170^2}=211{\cdot}8 \text{ amp}$$

$$i_2\ cos\phi_2=40, \text{ so } cos\phi_2=\frac{40}{i_2}=\frac{40}{211{\cdot}1}=0{\cdot}18$$

6. A circuit consisting of an inductance $\frac{2}{\pi}$ henry and a resistance of 103 ohms is connected to A.C. supply of 225volt-50 cycle. Calculate the power consumed and power factor.

Solution : $\quad Lp=2\pi.50.\frac{2}{\pi}=200$ ohms.

$R=103$ ohms.

$Z=\sqrt{R^2+L^2p^2}=\sqrt{200^2+103^2}=225$ ohms

Current $\quad i=\frac{E}{Z}=\frac{225}{225}=1$ amp.

Power factor $cos\phi=\frac{R}{Z}=\frac{103}{225}=0{\cdot}46$

Power$=Ei\ cos\phi=225\times1\times0{\cdot}46=103{\cdot}50$ watts.

VII-7. ELECTRICAL RESONANCE IN SERIES CIRCUIT

Condition for maximum current : When a source of alternating *emf* of pulsatance p is connected across a circuit containing a resistance R, a capacitance C and an inductance L in series, the impedance is given by

$$Z=\sqrt{R^2+(X_L-X_C)^2}=\sqrt{R^2+\left(Lp-\frac{1}{Cp}\right)^2}$$

The phase angle by which the applied *emf* leads the current is given by $\phi=tan^{-1}\left[\dfrac{Lp-\frac{1}{Cp}}{R}\right]$.

If $Lp>\frac{1}{Cp}$, ϕ is positive and the voltage leads the current. But if on the other hand $Lp<\frac{1}{Cp}$, ϕ becomes negative and the voltage lags behind the current. Further if $Lp=\frac{1}{Cp}$ *i.e.* $X_L=X_C$, Z becomes equal to R. The circuit then effectively behaves as a purely resistive circuit. This condition is known as *resonance.* At resonance the impedance Z is minimum and the current is in phase with the *emf* and its magnitude is maximum being obtained as $i=E_0 \sin pt/R$. It is the value of the current that would have been obtained in absence of L and C. The condition for maximum current or *current resonance*, as it is called, is

$$Lp=\frac{1}{Cp} \quad i.e. \quad p=\frac{1}{\sqrt{LC}} \quad \text{or } p^2LC=1.$$

This condition may otherwise be stated that for resonance the frequency of the applied *emf* should be $f=\frac{p}{2\pi}=\frac{1}{2\pi\sqrt{LC}}$. It is interesting to find that the natural frequency of an oscillatory circuit formed by an inductance L and capacitance C is also $\frac{1}{2\pi\sqrt{LC}}$ (§ VI-3). *Thus in L-R-C circuit current resonance*

occurs if the frequency of the applied emf is equal to the natural frequency of the corresponding $(L-C)$ *circuit.*

At resonance the voltage at the ends of the inductance (V_L) is equal and opposite to that between the capacitor plates (V_C). Since these are in opposite phase they always mutually cancel one another and as such they have no effect on the current in the circuit.

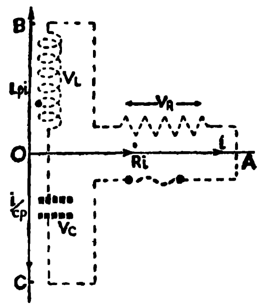


Fig. 7·13

For a resonant circuit (in which $X_L = X_c$) the current falls off with the deviation of the frequency of the applied *emf* from the natural frequency of the circuit. The response of the circuit to the applied *emf* being thus predominantly confined to a particular frequency, the circuit may be said to possess *selective property*. For this property such a circuit is called an *acceptor circuit* relating to a particular frequency. If the applied *emf* has a frequency other than the natural frequency of the circuit the impedance becomes considerably high so that the current falls to a low value. If the applied *emf* instead of having a single frequency is a complex one containing *emfs* of different frequencies, then a particular *L-R-C* circuit will give maximum response to the *emf* having the resonant frequency. In other words, the current produced in the circuit will be prominently of resonant frequency of the circuit. This is known as *selectivity*.

VOLTAGE MAGNIFICATION : At resonance the applied effective *emf* $E_{r.m.s.} = i_{r.m.s.} R$. This is so because the potential drops across the inductance and the capacitance being in anti-phase cancel each other. The applied *emf* overcomes only the ohmic resistance. The voltage at the ends of either inductance or capacitance is obtained in a magnified condition *i.e.* many times greater than the applied *emf*. The ratio of the potential drop across the inductance (which is same as that across the capacitance) and the applied *emf* is called the *magnification* denoted as *quality* or *Q-factor*. Hence

$$Q\text{-factor} = \frac{\text{P. D. across the inductance}}{\text{Applied } emf}$$

$$= \frac{Lpi_{r.m.s.}}{Ri_{r.m.s.}} = \frac{Lp}{R}$$

$$\text{Writing } p = \frac{1}{\sqrt{LC}},\ Q = \sqrt{\frac{L}{C}} \cdot \frac{1}{R}.$$

As expressed by Lp/R, Q-factor becomes the ratio of pulsatance p and the damping factor R/L or it is the ratio of reactance (of any component Lp or $1/Cp$) and the resistance. So for high value of Q, the circuit should have small resistance (R) and high value of ratio L/C. It may be noted that this magnified voltage does not cause any increase in power. It has no resultant effect on the circuit and exists individually and may reach values so as to cause damage to insulation. Hence resonance is to be avoided in industrial power circuits. On the other hand, it has important applications in radio circuits.

SHARPNESS OF RESONANCE : Resonance is said to be sharp when a slight deviation of the frequency of the applied *emf* from the resonant frequency of the circuit makes the current very low. Such sharpness depends upon the resistance of the circuit. For low values of R, the resonance becomes sharp, the selectivity is prominent. But for higher values of R, the response becomes flat over a wide range of frequencies and there is small variation of current with the deviation from resonant condition. The behaviour is shown in frequency-current curves. (Fig. 7·14-a).

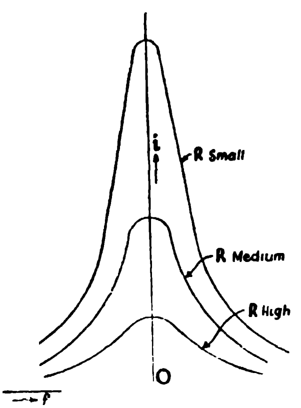


Fig. 7·14-a

Condition favouring sharp resonance may be realised from consideration of power dissipation. Let i_0 and i be respectively the current amplitudes at the resonant frequency p and at any

other frequency $(p+\delta p)$, slightly greater or less than p, δp may be either positive or negative. We have for resonance

$$i_o^2=\frac{E^2}{R^2+\left(Lp-\frac{1}{Cp}\right)^2}=\frac{E^2}{R^2}, \text{ since } Lp=\frac{1}{Cp}$$

For the other frequency slightly different we shall have

$$i^2=\frac{E^2}{\left[R^2+\left\{L(p+\delta p)-\frac{1}{C(p+\delta p)}\right\}^2\right]}$$

Now $$L(p+\delta p)-\frac{1}{C(p+\delta p)}=Lp+L.\delta p-\frac{1}{Cp}\left[1-\frac{\delta p}{p}-\cdots\right]$$

$$=Lp+L.\delta p-\frac{1}{Cp}+\frac{\delta p}{Cp^2}$$

$$=2L.\delta p, \text{ since } Lp=\frac{1}{Cp}$$

$$\text{and } \frac{\delta p}{p}\cdot\frac{1}{Cp}=L.\delta p$$

Hence $$i^2=\frac{E^2}{R^2.\left[1+\left(\frac{2L.\delta p}{R}\right)^2\right]}$$

Since power dissipation is proportional to the square of current, therefore

$$\frac{i_o^2}{i^2}=\frac{P_o}{P}=1+\frac{4L^2\delta p^2}{R^2}$$

P_o and P are the relevant power absorptions. Let us consider the case where this ratio of power absorption is halved due to a particular change δp in frequency.

So we may write,

$$1+\frac{4L^2\delta p^2}{R^2}=2$$

Hence $$\delta p=\frac{R}{2L}$$

Sharpness of resonance is determined by smallness of value of δp. Hence smaller the quantity R/L *i.e.* the damping coefficient, the sharper is the resonance.

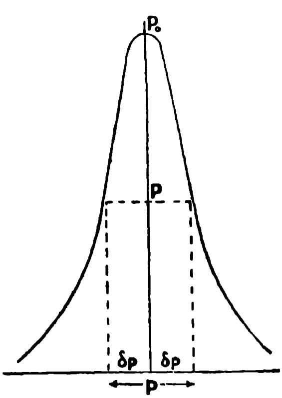


Fig. 7·14-b

Again, $\frac{p}{\delta p}=\frac{2Lp}{R}=2$ (Q-factor)

For better selectivity the ratio L/R should be large. Further, from the above equation the Q-factor may be defined as the ratio of the resonant frequency and the interval separating 'half-power' frequencies on either side.

Pass-band of a resonant circuit : For sharp resonance, the peak of the curve should be sharp. When $R=0$, the circuit has the extreme selectivity, allowing the current of one particular frequency to pass, the impedance to all other frequencies becomes infinite. When $R\neq 0$, the circuit allows current within a range of frequency, depending on the sharpness of the peak. This range of frequency is called the *Pass-band* of the resonant circuit.

Condition for maximum voltage : In a series L-R-C circuit the equation $p=1/\sqrt{LC}$ determines the condition for the maximum current. The condition for maximum voltage in the capacitance or at the ends of the inductance is not the same as that for maximum current. The voltage in the capacitance is given by

$$V_c = i.X_c = \frac{E.1/Cp}{\sqrt{R^2+\left(Lp-\frac{1}{Cp}\right)^2}}$$

$$\text{or}\quad V_c^2=\frac{E^2}{R^2C^2p^2+(p^2LC-1)^2}$$

For V_c to be maximum, the denominator of the above expression should be minimum. Denoting this as y, we consider the variation of y with C

$$y=R^2C^2p^2+(p^2LC-1)^2$$

For y to be maximum or minimum, $\frac{dy}{dC}=0$,

$$\text{or}\quad \frac{dy}{dC}=2CR^2p^2+2(p^2LC-1)\,p^2L=0$$

It is found that $\frac{d^2y}{dC^2}=2R^2p^2+2(p^2L)^2$ is positive

Hence $CR^2p^2+(p^2LC-1)p^2L=0$ corresponds to a minimum value of y. Hence the condition for maximum voltage drop in the capacitance in a series circuit is obtained as

$$C=\frac{L}{R^2+L^2p^2} \quad \text{or} \quad p=\frac{1}{\sqrt{LC-R^2/L^2}}.$$

It may be observed that this is different from the condition for maximum current (sometimes called current resonance) represented as $C=1/Lp^2$ or $p=1/LC$. The two conditions become identical when $R=0$..

By a similar treatment the condition for maximum voltage (sometimes called voltage resonance) at the ends of inductance may be obtained as

$$L=CR^2+\frac{1}{Cp^2} \quad \text{or} \quad p=\frac{1}{\sqrt{LC-C^2R^2}}$$

This also reduces to $L=\dfrac{1}{Cp^2}$ or $p=\dfrac{1}{\sqrt{LC}}$ when $R=0$.

It should be noted that series resonance causes *magnification* of voltage at the ends of inductance or capacitance in relation to applied *emf*, hence series resonance is called *voltage resonance* as distinguished from parallel resonance which causes current magnification and as such called *current resonance*.

VII·8. PARALLEL RESONANCE

Rejector circuit : When an inductance L of negligible resistance and a capacitance C are connected in parallel to an A.C. source, the current in the inductive branch lags and that in the capacitor branch leads the *emf* by $\pi/2$ and these are therefore in phase opposition and so the total current is the difference of the current in the two branches ($i_T=i_L\sim i_C$). It is either lagging or leading depending on the magnitude of reactances of the two branches. If under any condition the currents in the two branches are equal, the external current

becomes zero. In such a state though the source exists it does not send any current nor does it supply any energy. The current in the closed circuit comprising of the inductance and the capacitance if once initiated continues to oscillate at its natural pulsatance, *i.e.* $p=1/\sqrt{LC}$. Since there is no dissipation of energy in the process no further energy from source need be forthcoming. For this ideal state of affairs the the condition to be satisfied is given by

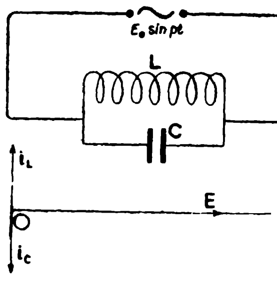


Fig. 7·15

$$i_L = i_C$$

$$\text{or} \quad \frac{E}{Lp} = \frac{E}{1/Cp}$$

$$\text{or} \quad p^2 = \frac{1}{LC}, \quad p = \frac{1}{\sqrt{LC}}$$

$$\text{or} \quad \text{frequency } f = \frac{1}{2\pi\sqrt{LC}}$$

This equation shows that the frequency (f) of the impressed *emf* should be equal to the natural frequency of the circuit.

In such an arrangement the external circuit impedance (Z_0) is maximum for the particular frequency as shown

$$Z_0 = \frac{E}{i} = \frac{E}{O} = \infty$$

That the equivalent impedance of the external circuit is infinite at resonance may also be realised by considering that X_L and X_C are in parallel in the circuit and these are equivalent to

$$Z_0 = \frac{X_L \cdot X_C}{X_L - X_C}, \quad (X_C \text{ is negative})$$

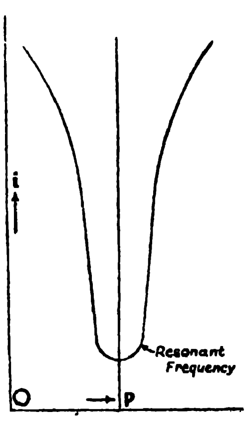


Fig. 7·16

$$\text{Since } X_L - X_C = Lp - \frac{1}{Cp} = 0, \text{ hence } Z_0 = \infty$$

For the resonant frequency the *external circuit current is minimum*. Such a circuit is called a *rejector circuit* for the *emf* of appropriate frequency. The fact may be used to filter out or reject the current of particular frequency.

PRACTICAL REJECTOR CIRCUIT : In practice, because of the existence of some resistance, however small, in the inductance, there is some dissipation of energy and to maintain the oscillations energy must be drawn from the source. The current i_z in the inductive branch (consisting of L and R) lags the *emf* by an angle ϕ $(\phi<\pi/2)$. The total current i_T represented as $\mathbf{i}_T=(\mathbf{i}_L+\mathbf{i}_R)+\mathbf{i}_C=(\mathbf{i}_z+\mathbf{i}_C)$ is minimum at resonance but it is never zero and it is in phase with the applied *emf*. Resonance occurs when $i_L=i_C$, and these two are mutually opposite in phase, so at resonance $i_T=i_R$ (see Fig. 7·17b).

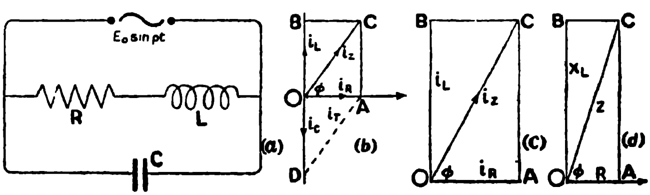


Fig. 7·17

Considering i_L and i_z as the currents in inductance proper and in the inductive branch (concerning X_L and R)

$$\mathbf{i}_z=\mathbf{i}_L+\mathbf{i}_R$$

and $i_L=i_z \sin\phi$ and $i_R=i_z \cos\phi$ (see Fig. 7·17c) at resonance $i_C=i_L=i_z \sin\phi$

But if Z is the impedance of the inductive branch,

$$i_z=\frac{E}{Z} \text{ and } \sin\phi = \frac{X_L}{Z} \text{ (Fig. 7·17d), so } i_L=\frac{E}{Z}\cdot\frac{X_L}{Z}$$

Again $i_0=\dfrac{E}{X_c}$, and since at resonance $i_L=i_0$

$$\text{so } \frac{E}{X_0}=\frac{E}{Z}\cdot\frac{X_L}{Z}$$

$$\text{or} \quad X_L . X_c = Z^2$$

$$\text{or} \quad Lp . \frac{1}{Cp} = L^2p^2 + R^2$$

$$\text{or} \quad \frac{L}{C} = L^2p^2 + L^2$$

$$\text{or} \quad p^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

So the resonant frequency when there is resistance in the inductive circuit is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

It is slightly less than the natural frequency of the pure L-C circuit.

Further, at resonance $i_R = i_T = i_z \cos\phi$

$$\text{or} \quad i_T = i_z . \frac{R}{Z} = \frac{E}{Z} \cdot \frac{R}{Z} = \frac{ER}{Z^2}$$

Impedance of the circuit $Z_0 = \dfrac{E}{i_T} = \dfrac{E.Z^2}{E.R} = \dfrac{Z^2}{R}$

$$\text{or} \quad Z_0 = \frac{L^2p^2 + R^2}{R} = R + \frac{L^2p^2}{R}$$

when $p = \dfrac{1}{\sqrt{LC}}$, $\quad Z_0 = R + \dfrac{L^2}{LCR} = R + \dfrac{L}{CR}$

If R is small, $Z_0 = \dfrac{L}{CR}$ (nearly)

Also the condition $p = \dfrac{1}{\sqrt{LC}}$ may be regarded as resonance condition when R is small. So Z_0 is the impedance of the entire circuit at resonance and it is of non-inductive nature, since the current i_T is in phase with the applied *emf*. Z_0 is proportional to $1/R$, where R is the actual ohmic resistance in the inductive branch. Z_0 is called the *dynamic resistance* of the circuit.

CURRENT MAGNIFICATION : The peak value of the current from the supply source at resonance is called the *make-up current* and if E_0 be the peak voltage, the make-up current is given by

$$\frac{E_0}{Z_0}=\frac{E_0}{L/CR}=\frac{E_0CR}{L}$$

The peak value of the oscillatory current in the L-C circuit, considering that in the capacitor branch, we get $i_c=E_0Cp$. Thus the Q-factor considered in this case as the current magnification is obtained as

$$Q\text{-factor}=\frac{\text{oscillatory current}}{\text{make up current}}=\frac{E_0Cp}{E_0CR/L}=\frac{Lp}{R}$$

The current in the closed L-C circuit is many times greater (in the ratio Lp/R) than the external circuit current and hence parallel resonance is distinguished as ***current resonance.***

It may be noted that the parallel rejector circuit at resonance shows current-magnification (in the capacitor or inductance) whereas the series acceptor circuit at resonance shows voltage magnification in the same ratio. It is for this difference in behaviour so far as the magnification effects are concerned the series and parallel resonances are distinguished as *voltage and current resononces* respectively.

Note : A confusion may arise regarding the terms voltage and current resonances. When these terms are used exclusively regarding series resonance, these have significances different from the case where the two are used to distinguish between series and parallel resonances.

In *series resonance* maximum current (current resonance) is obtained under a condition (regarding values of L, C, R) different from that for maximum (magnified) voltage (voltage resonance).

Again since in ***series resonance*** (in acceptor circuit) there is magnification of voltage at the capacitance or inductance this is called voltage resonance. On the other hand in ***parallel resonance*** (in rejector circuit) there is magnification of current in L-C circuit, so this is called current resonance.

ILLUSTRATIVE EXAMPLES

1. Find the resonant frequency of an acceptor circuit consisting of an inductance of 0·05 henry in series with a capacitor of 20 μF and resistance 10 ohms. Calculate the potential drop across the condenser and obtain the Q-factor. The supply voltage is 50 with appropriate frequency for resonance.

Solution : $f=\dfrac{1}{2\pi\sqrt{LC}}=\dfrac{1}{2\pi\sqrt{20\times10^{-6}\times0{\cdot}05}}=160$ *c.p.s.*

$X_C=X_L=2\pi fL=2\pi\times160\times0{\cdot}05=50{\cdot}24$ ohms.

$Z=R=10$ ohms, hence $i=\dfrac{E}{R}=\dfrac{50}{10}=5$ amps

$V_L=V_C=iX_C=5\times50{\cdot}24=251{\cdot}20$ volts

$$Q\text{-factor}=\frac{V_C}{E}=\frac{251{\cdot}20}{50}=5.$$

This shows that the voltage magnification is 5 times.

1-A. In the foregoing example if the resistance and inductance are included in the same coil, calculate the potential drop across the coil.

Solution : In this case $V_L > V_C$, because in this circuit the p.d. at the ends of the coil is the vector sum of the reactive and ohmic components of the p.d.

$$V_{LR}=iZ=i\sqrt{X_L^2+R^2}=5\sqrt{(50{\cdot}24)^2+10^2}=256 \text{ volts}$$

2. Obtain the voltage magnification at the terminals of a choke coil having an inductance 0·4 henry and resistance 5 ohms in series with a capacitor of 10 μF when an emf of 200 volts is applied at the resonant frequency.

Solution : $f=\dfrac{1}{2\pi\sqrt{LC}}=\dfrac{1}{2\pi\sqrt{0{\cdot}4\times10\times10^{-6}}}=80$ c.p.s.

Current $i=\dfrac{E}{R}=40$ amperes

$X_C=X_L=2\pi f\times0{\cdot}4=201$ ohms

$V_c=X_ci=201\times40=8040$ volts

Voltage magnification is $\dfrac{8040}{200}=40{\cdot}2$

3. *Find the resonant frequency of the rejector circuit having a capacitance of 10 μF in parallel with a coil of inductance 2·5 henry and 300 ohms resistance. Calculate the line current at resonance when a 50-volt supply is applied at resonant frequency.*

Solution : $p=2\pi f=\sqrt{\frac{1}{LC}-\frac{R^2}{L^2}}=\sqrt{\frac{10^6}{25}-\frac{300^2}{(2{\cdot}5)^2}}=120$

$$f=\frac{60}{\pi}=19{\cdot}1 \text{ cycles per second.}$$

$$i_T=i_s \cos\phi=i_s.\frac{R}{Z}=\frac{ER}{Z^2}=\frac{50\times 300}{L^2p^2+R^2}$$

or $$i_T=\frac{15\times 10^3}{(2{\cdot}5\times 120)^2+300^2}=\frac{15\times 10^3}{18\times 10^3}=\frac{5}{6} \text{ amp.}$$

VII-9. COMBINATION OF IMPEDANCES

Vector form of Kirchhoff's laws : The two laws regarding the distribution of current in a network are also applicable to alternating currents of same sinusoidal wave form and pulsatance in the modified forms stated below.

First Law : *In any network of conductors carrying alternating current the vector sum of the current vectors meeting at a point is zero.*

Second Law : *In any closed circuit in a network of conductors the vector sum of the vectors representing the potential drops is equal to the vector sum of the electromotive forces.*

Impedances in series : Let the impedances Z_1, Z_2, Z_3,··· be joined in series. Then if **i** be the current vector when the applied voltage is **E**, we have

$$\mathbf{E}=\mathbf{Z}_1\mathbf{i}+\mathbf{Z}_2\mathbf{i}+\mathbf{Z}_3\mathbf{i}+\cdots\cdots\cdots$$

If **Z** be the equivalent vector operator for impedances, then **E**=**Zi**, hence

$$\mathbf{Z}=\mathbf{Z}_1+\mathbf{Z}_2+\mathbf{Z}_3+\cdots\cdots\cdots$$

Impedances in parallel : Let the impedances Z_1, Z_2, Z_3 ··· be joined in parallel and let $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ ··· ··· be the current

vectors in them respectively. If **E** be the potential vector, this will be same at the ends of each impedance, hence·

$$\mathbf{E}=\mathbf{Z}_1\mathbf{i}_1=\mathbf{Z}_2\mathbf{i}_2=\mathbf{Z}_3\mathbf{i}_3$$

$$\text{So} \quad \mathbf{i}_1=\frac{\mathbf{E}}{\mathbf{Z}_1},\quad \mathbf{i}_2=\frac{\mathbf{E}}{\mathbf{Z}_2},\quad \mathbf{i}_3=\frac{\mathbf{E}}{\mathbf{Z}_3}$$

If **Z** be the equivalent impedance operator $\mathbf{E}=\mathbf{Z}\mathbf{i}$ and $\mathbf{i}=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3+\ldots\ldots$

$$\text{So,} \quad \frac{\mathbf{E}}{\mathbf{Z}}=\frac{\mathbf{E}}{\mathbf{Z}_1}+\frac{\mathbf{E}}{\mathbf{Z}_2}+\frac{\mathbf{E}}{\mathbf{Z}_3}+\ldots\ \ldots\ \ldots$$

$$\text{Hence} \quad \frac{1}{\mathbf{Z}}=\frac{1}{\mathbf{Z}_1}+\frac{1}{\mathbf{Z}_2}+\frac{1}{\mathbf{Z}_3}+\ldots\ \ldots\ \ldots$$

The reciprocal of impedance is called *admittance*. Denoting it by Y

$$\mathbf{Y}=\mathbf{Y}_1+\mathbf{Y}_2+\mathbf{Y}_3+\ldots\ \ldots$$

ADMITTANCE AND ITS COMPONENTS: Just as impedance has two components, resistance and reactance, so also has the admittance. These components are *conductance* and *susceptance*, indicated by g and b respectively. Just as vector sum of **X** and **R** is obtained by drawing the impedance triangle, admittance triangle gives the relationship involved in g, b and Y. From the triangles shown in Fig. 7·18,

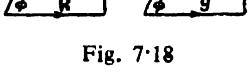


Fig. 7·18

$$b^2+g^2=Y^2=\frac{1}{Z^2}=\frac{1}{X^2+R^2}$$

$$\tan\phi=\frac{b}{g}=\frac{X}{R}$$

$$\text{Power factor } \cos\phi=\frac{g}{Y}$$

$$\text{and } \sin\phi=\frac{b}{Y}$$

Expressions for g and b: Since $\frac{b}{g}=\frac{X}{R}$

$$\frac{b^2}{b^2+g^2}=\frac{X^2}{X^2+R^2}$$

$$\text{or} \quad b^2=(b^2+g^2).\frac{X^2}{X^2+R^2}=\frac{1}{Z^2}.\frac{X^2}{Z^2}=\frac{X^2}{Z^4}$$

$$\text{Hence } b = \frac{X}{Z^2}$$

$$\text{Again, } \frac{b^2+g^2}{g^2} = \frac{X^2+R^2}{R^2}$$

$$\text{or } g^2 = R^2 . \frac{b^2+g^2}{X^2+R^2} = R^2 . \frac{1}{Z^2 . Z^2} = \frac{R^2}{Z^4}$$

$$\text{Hence } g = \frac{R}{Z^2}$$

ILLUSTRATIVE EXAMPLES

1. A 200-ohm resistance coil has in parallel an inductance of 4/π henry. The combination is connected to a 200-volt (r.m.s.) at 50-cycle supply. Draw a vector diagram showing the distribution and calculate the currents.

Solution : In the vector diagram OE represents the supply voltage. The current in the resistor i_R is in phase with the voltage and so it is represented by OA in the same direction as OE. The current in the inductance is $\pi/2$ behind the applied *emf* and is represented by OB in a clockwise direction from and at right angles to OE. Vector OC represents the supply current.

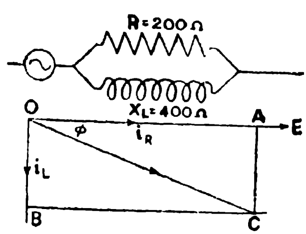


Fig. 7·19

$$Lp = \frac{4}{\pi} \times 2\pi \times 50 = 400 \text{ ohms}$$

$$i_R = \frac{200}{200} = 1 \text{ amp.}$$

$$i_L = \frac{200}{400} = 0{\cdot}5 \text{ amp.}$$

$$i = \sqrt{i_R^2 + i_L^2} = \sqrt{1^2 + (0{\cdot}5)^2} = 1{\cdot}11 \text{ amps.}$$

The main current lags the *emf* by $\phi = \tan^{-1}\left(\frac{0{\cdot}5}{1}\right) = 26^\circ{\cdot}6.$

2. *Two coils, one of resistance 8 ohms and inductance 0·0191 henry and the other of resistance 9 ohms and inductance 0·0382 henry, are arranged in parallel circuit with a supply of 100 volt-50 cycle. Find the current in the circuit.*

Solution : Such problems may be solved by different methods. These are shown separately.

Relevant data are obtained as shown.

In the branch-1, $R_1=8\ \Omega$, $L=0{\cdot}0191\ H$, $E=100$ volts

$$X_1=L_1p=0{\cdot}0191\times2\times3{\cdot}14\times50=6\ \Omega$$

$$Z_1=\sqrt{R_1{}^2+X_1{}^2}=\sqrt{8^2+6^2}=10\ \Omega$$

In the branch-2, $R_2=9\ \Omega$, $L=0{\cdot}0382\ H$

$$X_2=L_2p=0{\cdot}0382\times2\times3{\cdot}14\times50=12\ \Omega$$

$$Z_2=\sqrt{R_2{}^2+X_2{}^2}=\sqrt{9^2+12^2}=15\ \Omega$$

(a) SOLUTION BY SYMBOLIC METHOD :

Applying $$i=\frac{E}{R+jLp}$$

$$i_1=\frac{100}{8+6j}=\frac{100(8-6j)}{8^2+6^2}=8-6j$$

Hence $i_1=\sqrt{8^2+6^2}=10$ amps.

Phase $\phi_1=\tan^{-1}\left(\frac{6}{8}\right)=36^\circ{\cdot}9$ (lagging)

$$i_2=\frac{100}{9+12j}=\frac{100(9-12j)}{9^2+12^2}=4-\frac{16}{3}j$$

Hence $i_2=\sqrt{4^2+\left(\frac{16}{3}\right)^2}=6{\cdot}6$ amps.

Phase $\phi_2=\tan^{-1}\left(\frac{16}{12}\right)=53^\circ{\cdot}1$ (lagging)

Total current $i=i_1+i_2=(8-6j)+\left(4-\frac{16}{3}j\right)=12-\frac{34}{3}j$

Hence $i=\sqrt{12^2+\left(\frac{34}{3}\right)^2}=16{\cdot}7$ amps.

Phase $\phi=tan^{-1}\left(\frac{34}{36}\right)=43^\circ{\cdot}3$ (lagging)

(b) SOLUTION BY METHOD OF RESOLUTION : The branch currents and their phases are obtained by methods shown in (a). These currents are resolved along the direction of *emf* (active component) and at right angles to it (reactive component).

$$\text{Active components} = i_1 \cos\phi_1 + i_2 \cos\phi_2 = i \cos\phi$$

$$\text{or} \quad i \cos\phi = 10 \times \cos 36^\circ{\cdot}9 + 6{\cdot}6 \times \cos 53^\circ{\cdot}1$$

$$\text{or} \quad i \cos\phi = 10 \times 0{\cdot}7997 + 6{\cdot}6 \times 0{\cdot}6004 = 11{\cdot}959$$

$$\text{Reactive components} = i_1 \sin\phi_1 + i_2 \sin\phi_2 = i \sin\phi$$

$$\text{or} \quad i \sin\phi = 10 \times \sin 36^\circ{\cdot}9 + 6{\cdot}6 \times \sin 53^\circ{\cdot}1$$

$$\text{or} \quad i \sin\phi = 10 \times 0{\cdot}6004 + 6{\cdot}6 \times 0{\cdot}7997 = 11{\cdot}282$$

$$\text{Main current} \quad i = \sqrt{i^2\cos^2\phi + i^2 \sin^2\phi} = \sqrt{(11{\cdot}959)^2 + (11{\cdot}282)^2}$$

$$\text{or} \quad i = 16{\cdot}7 \text{ amps.}$$

$$\text{Phase } \phi = \tan^{-1}\left[\frac{i \sin\phi}{i \cos\phi}\right] = \tan^{-1}\left[\frac{11{\cdot}282}{11{\cdot}959}\right] = 43^\circ{\cdot}3 \text{ (lagging)}$$

This method applies for any number of branches in parallel.

(c) SOLUTION BY ADMITTANCE METHOD : Applying $g = \frac{R}{Z^2}$, $b = \frac{X}{Z^2}$ and $\phi = \tan^{-1}\left(\frac{b}{g}\right)$, we get

$$g_1 = \frac{R_1}{Z_1^2} = \frac{8}{100} = 0{\cdot}08 \text{ mho.}$$

$$b_1 = \frac{X_1}{Z_1^2} = \frac{6}{100} = 0{\cdot}06 \text{ mho.}$$

$$Y_1 = \sqrt{g_1^2 + b_1^2} = \sqrt{({\cdot}08)^2 + ({\cdot}06)^2} = 0{\cdot}1 \text{ mho.}$$

$$\text{Hence} \quad i_1 = 100 \times 0{\cdot}1 = 10 \text{ amps.}$$

$$\tan\phi_1 = \frac{b_1}{g_1} = \frac{6}{8} \quad \text{whence } \phi_1 = 36^\circ{\cdot}9$$

$$g_2 = \frac{R^2}{Z_2^2} = \frac{9}{225} = 0{\cdot}04 \text{ mho}$$

$$b_2 = \frac{X_2}{Z_2^2} = \frac{12}{225} = 0{\cdot}053 \text{ mho}$$

$$Y_2 = \sqrt{g_2^2 + b_2^2} = \sqrt{({\cdot}04)^2 + ({\cdot}053)^2} = 0{\cdot}066 \text{ ohm}$$

Hence $i_2 = 100 \times 0{\cdot}066 = 6{\cdot}6$ amps.

Total conductance $g = g_1 + g_2 = 0{\cdot}12$ mho

Total susceptance $b = b_1 + b_2 = 0{\cdot}113$ mho

Total admittance $Y = \sqrt{g^2 + b^2} = 0{\cdot}167$ mho

Total current $i = 100Y = 16{\cdot}7$ amps.

Phase $\phi = \tan^{-1}\left(\frac{b}{g}\right) = \tan^{-1}\left(\frac{113}{120}\right) = 43^\circ{\cdot}3$ (lagging)

(d) SOLUTION BY IMPEDANCE METHOD :

Applying $i = \frac{E}{Z}$,

$$i_1 = \frac{E}{Z_1} = \frac{100}{10} = 10 \text{ amps}, \quad \phi_1 = \tan^{-1}\left(\frac{6}{8}\right) = 36^\circ{\cdot}9$$

$$i_2 = \frac{E}{Z_2} = \frac{100}{15} = 6{\cdot}6 \text{ amps}, \quad \phi_2 = \tan^{-1}\left(\frac{12}{9}\right) = 53^\circ{\cdot}1$$

$$i = \sqrt{i_1^2 + i_2^2 + 2\cos(\phi_1 - \phi_2) i_1 i_2}$$

or $i = \sqrt{10^2 + (6{\cdot}6)^2 + 2.10.6{\cdot}6 \cos 16^\circ{\cdot}2} = 16{\cdot}7$ amps.

$$\phi = \tan^{-1}\left[\frac{i_2 \sin(\phi_2 - \phi_1)}{i_1 + i_2 \cos(\phi_2 - \phi_1)}\right] + \phi_1 = 6^\circ{\cdot}4 + 36^\circ{\cdot}9$$

or $\phi = 43^\circ{\cdot}3$ (lagging)

3. *What voltage should be applied to produce a total current of 4 amperes in two coils in parallel, the coils having the following specifications.*

(i) $R_1 = 3$ ohms, $X_1 = 4$ ohms (ii) $R_2 = 6$ ohms, $X_2 = 8$ ohms.

Solution by Method-1: $Z_1 = \sqrt{R_1^2 + X_1^2} = \sqrt{3^2 + 4^2} = 5$ ohms.

$$\phi_1 = \tan^{-1}\left(\frac{X_1}{R_1}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 36^\circ{\cdot}9$$

$$Z_2 = \sqrt{R_2^2 + X_2^2} = \sqrt{6^2 + 8^2} = 10 \text{ ohms.}$$

$$\phi_2 = \tan^{-1}\left(\frac{X_2}{R_2}\right) = \tan^{-1}\left(\frac{8}{6}\right) = 36^\circ{\cdot}9$$

The currents in the two branches are in phase. Since $\phi_1=\phi_2$ so total current $i=i_1+i_2=\frac{E}{Z_1}+\frac{E}{Z_2}$

$$\text{or} \quad 4=E\left(\frac{1}{5}+\frac{1}{10}\right)=\frac{3E}{10}$$

$$\text{or} \quad E=\frac{40}{3}=13{\cdot}33 \text{ volts}$$

Solution by Method-2 :

$$\text{Conductance} \quad g_1=\frac{R_1}{Z_1^2}=\frac{R_1}{X_1^2+R_1^2}=\frac{3}{4^2+3^2}=\frac{3}{32}=0{\cdot}12$$

$$\text{Conductance} \quad g_2=\frac{R_2}{Z_2^2}=\frac{R_o}{X_2^2+R_2^2}=\frac{6}{8^2+6^2}=\frac{6}{100}=0{\cdot}06$$

Total conductance $g=g_1+g_2=0\ 12+0{\cdot}06=0{\cdot}18$ mho

$$\text{Susceptance} \quad b_1=\frac{X_1}{Z_2^2}=\frac{4}{25}=0{\cdot}16$$

$$\text{Susceptance} \quad b_2=\frac{X_2}{Z_2^2}=\frac{8}{100}=0{\cdot}08$$

Total susceptance $b=b_1+b_2=0{\cdot}16+0{\cdot}8=0{\cdot}24$ mho

Total admittance $Y=\sqrt{g^2+b^2}=\sqrt{(0{\cdot}18)^2+(0{\cdot}24)^2}=0{\cdot}3$ mho

$$\text{Voltage} \quad E=\frac{i}{Y}=\frac{4}{0{\cdot}3}=13{\cdot}33 \text{ volts.}$$

4. A circuit consists of three branches is parallel having the following specifications :

(i) $R_1=12$ ohms, $L_1=0{\cdot}0159$ herry (ii) $R_2=5$ ohms, $C_2=100\ \mu F$ (iii) $R_3=10$ ohms.

Find the total current and the phase angle when the supply voltage is 200 V-50 cycle.

Solution by Method-1 :

(i) $R_1=12\ \Omega,\ L_1=0{\cdot}0159\ H,\ X_1=L_1p=0{\cdot}159\times 314=5\ \Omega$

$$Z_1=\sqrt{12^2+5^2}=13\ \Omega$$

$$cos\phi_1=\frac{R_1}{Z_1}=\frac{12}{13},\ sin\phi_1=\frac{X_1}{Z_1}=\frac{5}{13}.$$

$$i=\frac{200}{13}=15{\cdot}4,\ i_1cos\phi_1=14{\cdot}21,\ i_1\ sin\phi_1=5{\cdot}9.$$

(ii) $R_2 = 5\Omega$, $C = 100\times10^{-6}F$,

$$X_2 = -\frac{1}{Cp} = -\frac{10^6}{314\times100} = -31{\cdot}84\ \Omega$$

$$Z_2 = \sqrt{5^2+(31{\cdot}84)^2} = 32{\cdot}2\ \Omega$$

$$\cos\phi_2 = \frac{5}{32{\cdot}2} \text{ and } \sin\phi_2 = -\frac{31{\cdot}84}{32{\cdot}2}$$

$i_2 \cos\phi_2 = 0{\cdot}964$ and $i_2 \sin\phi_2 = -6{\cdot}14$

(iii) $R_3 = 10\ \Omega$, $i_3 = \frac{200}{10} = 20$ amp, $\cos\phi_3 = 1$, $\sin\phi_3 = 0$

$$i\cos\phi = i_1\cos\phi_1 + i_2\cos\phi_2 + i_3\cos\phi_3$$
$$= 14{\cdot}26 + 0{\cdot}964 + 20 = 35{\cdot}174$$

$$i\sin\phi = i_1\sin\phi_1 + i_2\sin\phi_2 + i_3\sin\phi_3 = 5{\cdot}9 - 6{\cdot}14 = -0{\cdot}24$$

Hence $i = \sqrt{(35\ 174)^2+(0{\cdot}24)^2} = 35{\cdot}2$ amp.

$$\phi = \tan^{-1}\left[\frac{-0{\cdot}24}{35{\cdot}174}\right] = 0^\circ{\cdot}4 \text{ (current leading)}$$

Solution by method-2 :

(i) $R_1 = 12\ \Omega$, $X_1 = Lp = 0{\cdot}0159\times314 = 5\ \Omega$

$Z_1 = R_1 + jX_1 = 12+5j$ ohms

So $i_1 = \frac{200}{12+5j} = \frac{200(12-5j)}{12^2+5^2} = \frac{200}{169}(12-5j)$

or $i_1 = 14{\cdot}2 - 5{\cdot}9j$

(ii) $R_2 = 5\ \Omega$, $X_2 = -j/Cp = -\frac{10^6 j}{31400} = -31{\cdot}84j$

$Z_2 = 5 - 31{\cdot}84j$

$$i_2 = \frac{200}{5-31{\cdot}84j} = \frac{200(5+31{\cdot}84j)}{1038} = 1+6{\cdot}14j \text{ amp}$$

(iii) $R_3 = 10\Omega$, $i_3 = \frac{200}{10} = 20$ amps.

Total current $i = i_1+i_2+i_3 = 14{\cdot}2 - 5{\cdot}9j + 1 + 6{\cdot}14j + 20$

or $i = 35{\cdot}2 + 0{\cdot}24j$

Magnitude of $i = \sqrt{(35{\cdot}2)^2+(0{\cdot}24)^2} = 35{\cdot}2$ amp.

Phase $\phi = \tan^{-1}\left[\frac{0{\cdot}24}{35{\cdot}2}\right] = 0^\circ{\cdot}4$

(current leads since j-component is positive

5. *A 200 volt-50 cycle supply is connected across a series circuit cotaining an inductive resistance of 0 5 henry and 10 ohms, a condenser of 100 μF capacitance and an ohmic resistance of 5 ohms. Calculate the current and the potential drop across the inductive resistance coil.*

Solution : Impedance $Z=\sqrt{(R_1+R_2)^2+\left(Lp-\frac{1}{Cp}\right)^2}$

So $Z=\sqrt{(10+5)^2+\left(314\times0{\cdot}5-\frac{10^4}{314}\right)^2}$

or $Z=\sqrt{15^2+(16{\cdot}14)^2}=22{\cdot}3$ ohms.

Hence $i=\frac{E}{Z}=\frac{200}{22{\cdot}3}=9$ amps (nearly)

Impedance of the inductive coil $Z_1=\sqrt{10^2+(15{\cdot}7)^2}=18{\cdot}6$ ohms

P.D. across $Z_1=iZ_1=9\times18{\cdot}6=167{\cdot}4$ volts

$$\phi=tan^{-1}\left(\frac{15{\cdot}7}{10}\right)=57^\circ{\cdot}5 \text{ (lagging)}$$

6. *Two inductive resistance coils are connected in series with a supply of A.C. whose peak-voltage is 353·5 volts at 50 c.p.s. If one coil has 5 ohms resistance and 0·03 henry inductance and the other coil 6 ohms resistance and 0·0241 henry inductance, calculate the effective current through the coils, the total power consumed and the phase angle.*

What additional component will make the current maximum ?

Solution : $R_1=5\ \Omega,\ X_1=0{\cdot}03\times2\pi\times50=9{\cdot}43\ \Omega$

$R_2=6\ \Omega,\ X_2=0{\cdot}024\times2\pi\times50=7{\cdot}57\ \Omega$

$R=R_1+R_2=11\ \Omega,\ X=X_1+X_2=17\ \Omega$

$Z=\sqrt{11^2+17^2}=20\ \Omega$

$$tan\ \phi=\frac{X}{R}=\frac{17}{11},\ \phi=57^\circ{\cdot}2$$

$$cos\ \phi=\frac{R}{Z}=\frac{11}{20}=0{\cdot}5417$$

R.M.S. value of *emf*, $E=\frac{E_0}{\sqrt{2}}=\frac{353{\cdot}5}{\sqrt{2}}=250$ volts

Effective current $i = \frac{E}{Z} = \frac{250}{20} = 12 \cdot 5$ amps.

Power consumed $P = Ei \cos \phi = 250 \times 12 \cdot 5 \times 0 \cdot 54$

or $P = 1687 \cdot 5$ watts.

A capacitor of such value as to produce resonance joined in series will make the current maximum.

For resonance $X_L = X_C$ *i.e.* $\frac{1}{Cp} = 17$ ohms

Hence $C = \frac{1}{17 \times 100\pi}$

or $C = 0 \cdot 0187$ farad.

VII-10. A.C. BRIDGES

Wheatstone's net : Let $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$ be the impedances of the four branches indicated in diagram (Fig. 7·20). The supply is from an A.C. source and the detector is either a telephone receiver or a vibration galvanometer. Impedances in these two branches are respectively $\mathbf{Z}_5$ and $\mathbf{Z}_6$.

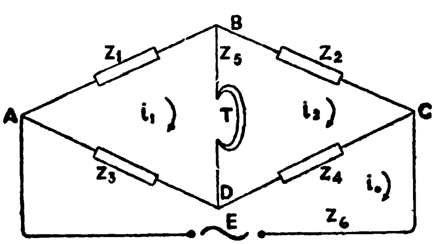


Fig. 7·20

Let the instantaneous *cyclic currents* in the three meshes as shown be $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_0$. The current through the detector at any instant is $\mathbf{i} = \mathbf{i}_1 - \mathbf{i}_2$.

Applying Maxwell's modified form of Kirchhoff's law, we get

In the mesh *ABD*, $\mathbf{Z}_1\mathbf{i}_1 + \mathbf{Z}_5(\mathbf{i}_1 - \mathbf{i}_2) + \mathbf{Z}_3(\mathbf{i}_1 - \mathbf{i}_0) = 0$

In the mesh *BCD*, $\mathbf{Z}_2\mathbf{i}_2 + \mathbf{Z}_4(\mathbf{i}_2 - \mathbf{i}_0) + \mathbf{Z}_5(\mathbf{i}_2 - \mathbf{i}_1) = 0$

Rearranging terms,

$$-\mathbf{Z}_3\mathbf{i}_0 + (\mathbf{Z}_1 + \mathbf{Z}_5 + \mathbf{Z}_3)\,\mathbf{i}_1 - \mathbf{Z}_5\mathbf{i}_2 = 0$$

and $$-\mathbf{Z}_4\mathbf{i}_0 - \mathbf{Z}_5 i_1 + (Z_2 + \mathbf{Z}_4 + \mathbf{Z}_5)\,\mathbf{i}_2 = 0$$

For eliminating $\mathbf{i}_0$,

$$-\mathbf{Z}_3\mathbf{Z}_4\mathbf{i}_0 + (\mathbf{Z}_1 + \mathbf{Z}_5 + \mathbf{Z}_3)\,\mathbf{Z}_4\mathbf{i}_1 - \mathbf{Z}_5\mathbf{Z}_4\mathbf{i}_2 = 0$$

and $$-\mathbf{Z}_3\mathbf{Z}_4\mathbf{i}_0 + \mathbf{Z}_5\mathbf{Z}_3\mathbf{i}_1 + (\mathbf{Z}_2 + \mathbf{Z}_4 + \mathbf{Z}_5)\mathbf{Z}_3\mathbf{i}_2 = 0$$

By subtraction,

$$[(\mathbf{Z}_1+\mathbf{Z}_3+\mathbf{Z}_5)\mathbf{Z}_4+\mathbf{Z}_5\mathbf{Z}_3]\mathbf{i}_1=[\mathbf{Z}_4\mathbf{Z}_5+(\mathbf{Z}_2+\mathbf{Z}_4+\mathbf{Z}_5)\mathbf{Z}_3]\mathbf{i}_2$$

For no current in the detector $\mathbf{i}_1=\mathbf{i}_2$

Hence $(\mathbf{Z}_1+\mathbf{Z}_3+\mathbf{Z}_5)\mathbf{Z}_4+\mathbf{Z}_5+\mathbf{Z}_3=(\mathbf{Z}_2+\mathbf{Z}_4+\mathbf{Z}_5)\mathbf{Z}_3+\mathbf{Z}_4\mathbf{Z}_5$

or $\mathbf{Z}_1\mathbf{Z}_4=\mathbf{Z}_2\mathbf{Z}_3$.

Expressed in the symbolic form

$$Z_1Z_4e^{j(\phi_1+\phi_4)}=Z_2Z_3e^{j(\phi_2+\phi_3)},$$

ϕ_1, ϕ_2, ϕ_3 and ϕ_4 being the respective phase angles.

When the two vector expressions on the two sides are equal, they have equal magnitudes and they are coincident in phase.

The necessary conditions are obtained by equating the real and imaginary parts of the above equation, *i.e.*

$$Z_1Z_4 = Z_2Z_3$$

$$\text{and} \quad (\phi_1+\phi_4) = (\phi_2+\phi_3)$$

These are to be satisfied for a balance.

Owen's bridge for measurement of self-inductance : The bridge is arranged as shown in the diagram (Fig. 7·21). L is the self-inductance having ohmic resistance r, C_3 and C_4 are two capacitors, R_1 and R_2 are two resistances.

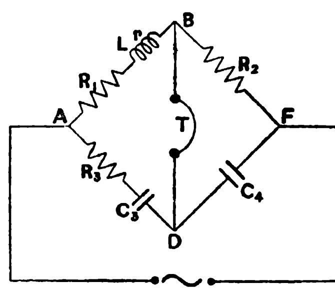


Fig. 7·21

Applying the condition for null in the wheatstone bridge when the bridge is balanced

$$\frac{R_1+r+jLp}{R_2}=\frac{R_3-\dfrac{j}{C_3p}}{-\dfrac{j}{C_4p}}$$

$$\text{or} \quad \frac{-j(R_1+r)}{C_4p}+\frac{L}{C_4}=R_3R_2-\frac{jR_2}{C_3p}$$

Equating the real and imaginary parts, we obtain the necessary conditions for balance

$$L=R_3R_2C_4 \quad \text{and} \quad (R_1+r)C_3=R_2C_4$$

If $C_3=C_4=C$ (say), the two conditions reduce to

$$L=CR_2R_3 \quad \text{and} \quad r=R_2-R_1$$

The two conditions may be satisfied independent of each other. The minimum response in the detector should be obtained by

varying R_1. This should then be reduced by varying R_3. R_1 and R_3 should be adjusted alternately until the null is obtained.

This bridge is suitable for measurement of self-inductance from a few micro-henries upwards. The sensitivity is very high and the balance is independent of frequency and hence of the wave form of the applied voltage.

Maxwell's bridge for comparison of self-inductances : The arrangement is as shown in the diagram (Fig. 7·22). L_1 and L_2 are inductances to be compared and R_1 and r are respectively their resistances.

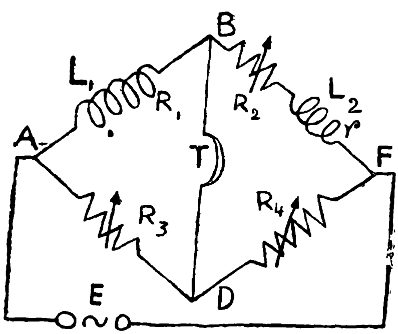


Fig. 7·22

R_2, R_3, R_4 are non-inductive variable resistances. For a balanced bridge

$$\frac{R_1+jL_1p}{R_3}=\frac{(R_2+r)+jL_2p}{R_4}$$

or $R_1R_4+jL_1pR_4$

$$=R_3R_2+R_3r+jL_2pR_3$$

Equating real and imaginary parts,

$$\frac{R_1}{R_2+r}=\frac{R_3}{R_4} \text{ and } \frac{L_1}{L_2}=\frac{R_3}{R_4}$$

The bridge is not suitable for practical use since the two conditions are not independent of one another.

Anderson's bridge for measurement of self-inductance : The bridge is not a true wheatstone bridge. The circuit is shown in Fig. 7·23. There are four resistances in the four arms as in wheatstone bridge and the inductance L is in series with the resistance R_1, in one of the arms. A condenser C in series with a non-inductive resistance r is placed in parallel with the fourth arm resistance R_4.

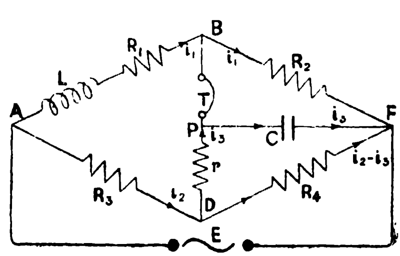


Fig. 7·23

Let the current in different branches under the null condition be as indicated in

the diagram. Applying Kirchhoff's law,

In the mesh $ABPD$: $(R+jLp)i_1-ri_3-R_3i_2 \quad =0$ (i)

In the mesh BFD : $R_2i_1-i_3/jCp \quad =0$ (ii)

In the mesh DFP : $\left(r+\dfrac{1}{jCp}\right)i_3-R_4(i_2-i_3)=0$ (iii)

Eliminating i_3 from (ii) and (iii)

$$\left(r+\frac{1}{jcp}+R_4\right)R_2\,jCpi_1 \;=\; R_4i_2 \qquad \text{(iv)}$$

Eliminating i_3 from (i) and (ii)

$$[R+jLp-jCprR_2]i_1 \;=\; R_3i_2 \qquad \text{(v)}$$

Eliminating i_1 and i_2 from (iv) and (v)

$$jCpr\;R_2R_3+R_2R_3+jCp\;R_2R_3R_4=R_1R_4+jpLR_4-jCprR_2R_4$$

Equating real and imaginary parts, we get

$$R_1R_4=R_2R_3$$

$$\text{and } LR_4-rR_2R_4C=rR_2R_3C+R_2R_3R_4C$$

$$\text{Hence}\quad L=CrR_2+\frac{CrR_2R_3}{D}+R_2R_3C$$

$$\text{or}\quad L=CR_2\left[r\left(1+\frac{R_3}{R_4}\right)+R_3\right]$$

The two conditions may be satisfied independently. The bridge is adjusted to the balanced condition by alternately varying r and R_1. The double balance is obtained only by adjusting the resistances, the condenser may be a standard one of a fixed value.

Maxwell's Inductance-Capacitance bridge : In this arrangement an inductance is measured in terms of a standard capacitance and two resistances. Three non-inductive resistances are taken and are arranged to form three arms as R_2, R_3, R_4 (Fig. 7·24). The inductance L in series with a variable resistance R_1 is placed in the remaining arm. The capacitance C is in parallel with R_4. The capacitance is variable. When the bridge is balanced, the relation satisfied is

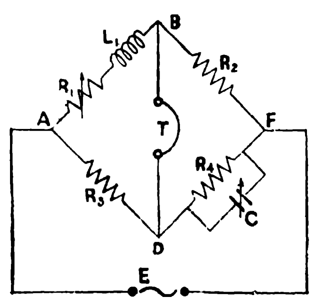


Fig. 7·24

$$\frac{R_1+jLp}{R_3}=R_2\Big/\frac{R_4}{1+jR_4Cp}$$

or $$\frac{R_1+jLp}{R_3}=\frac{R_2}{R_4}(1+jR_4Cp)$$

or $$R_1R_4+jLpR_4=R_3R_2+jR_2R_3R_4Cp$$

Equating real and imaginary parts, we get

$$\frac{R_1}{R_3}=\frac{R_2}{R_4} \text{ and } \frac{L}{R_3}=CR_2$$

$$\text{Hence } L=CR_2R_3$$

R_2 : R_3 is kept constant as a simple ratio and R_1 is varied to obtain a minimum current (in phase) through the detector. Then C is varied to obtain a minimum again for quadrature. Two or three adjustments may be necessary for final balance.

de Sauty's bridge for comparison of capacitances : The bridge is a wheatstone bridge with the modification that the resistors in the second and fourth arms are replaced by capacitors to be compared. When the bridge is balanced the condition satisfied should be

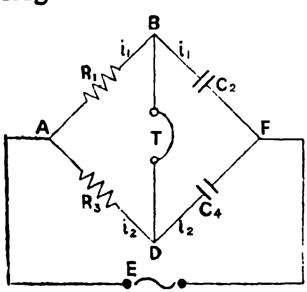


Fig. 7·25

$$\frac{R_1}{R_3}=\frac{-j/C_2p}{-j/C_4p}=\frac{C_4}{C_2}$$

The condition is obtained by considering the potential drops in the circuit. The potential difference between A-B is same as that between A-D under null conditions, so

$$R_1i_1=R_3i_2 \quad \text{or} \quad \frac{i_1}{i_2}=\frac{R_3}{R_1}$$

Again the potential difference between B-F and that between D-F are equal, so

$$-\frac{ji_1}{C_2p}=-\frac{ji_2}{C_4p} \quad \text{or} \quad \frac{i_1}{i_2}=\frac{C_2}{C_4}$$

Hence $$\frac{R_1}{R_3}=\frac{C_4}{C_2}$$

For an *imperfect condenser*, while considering the impedance the capacitance should be considered to be in series with a resistance r. If in de Sauty's bridge the condensers are imperfect, the condition for null should be written as,

$$\frac{R_1}{R_3}=\frac{r_2-j/C_2p}{r_4-j/C_4p}.$$

$$\text{or}\quad R_1r_4-\frac{jR_1}{C_4p}=R_3r_2-\frac{jR_3}{C_2p}.$$

Equating real and imaginary parts

$$\frac{r_2}{r_4}=\frac{R_1}{R_3}=\frac{C_4}{C_2}$$

Balance in such a bridge is obtained with difficulty.

Schering bridge for comparison of capacitances : The bridge is arranged as shown in the diagram (Fig. 7·26). The condenser under test (C_1r) is in the first arm and in the second and the third arms there are respectively the resistances R_2 and the standard condenser C_3. In the fourth arm a variable air condenser is in parallel with a variable resistance R_4. When the bridge is balanced

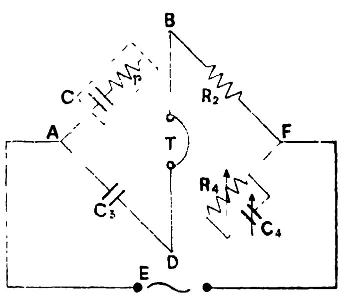


Fig. 7·26

$$\frac{r+1/jCp}{1/jC_3p}=\frac{R_2}{Z_4}$$

$$\text{where}\quad \frac{1}{Z_4}=\frac{1}{R_4}+jpC_4.$$

$$\text{Hence,}\quad \frac{r+1/jCp}{1/jC_3p}=R_2\left(\frac{1}{R_4}+jpC_4\right)$$

$$\text{or}\quad r+1/jCp=\frac{R_2}{jR_4C_3p}+\frac{R_2C_4}{C_3}$$

Equating real and imaginary parts,

$$r=\frac{R_2C_4}{C_3}\quad\text{and}\quad\frac{1}{C}=\frac{R_2}{C_3R_4}\quad\text{or}\quad C=\frac{C_3R_4}{R_2}$$

To obtain the power factor $\cos\phi$ of the condenser C, it may be noted that in such a case since r is the leakage resistance, as shown in § VII-6, $\cos\phi = Cpr$, so

$$\cos\phi = Cpr = p.\frac{R_2C_4}{C_3}.\frac{C_3R_4}{R_2}$$

$$\text{or} \quad \cos\phi = pC_4R_4.$$

Schering bridge is the most accurate method for determination of capacitance relative to a standard capacitor. It is used widely for the measurement of dielectric constant specially of liquids and for determination of the value of small capacitances.

Campbell bridge for measurement or Mutual Inductance: Carey Foster's D.C. method (§ V-7) for measurement of mutual inductance was modified by Heydweiller and Campbell as an A.C. method. The arrangement is as shown in the diagram (Fig. 7·27). Considering the flow of current in different branches at a stage when the bridge is balanced, we have if M is the mutual inductance,

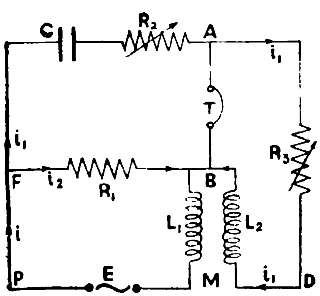


Fig. 7·27

$$\text{at } F,\ i = i_1 + i_2 \quad \dots \quad \text{(i)}$$

In the mesh ABD:

$$jpMi - jpL_2i_1 - i_1R_3 = 0 \quad \dots \quad \text{(ii)}$$

In the mesh BAF: $i_2R_1 - i_1R_2 - i_1/jpC = 0$ ... (iii)

$$\text{From (ii)} \quad i_1 = \frac{jpMi}{R_3 + jpL_2}$$

$$\text{From (i)} \quad i_2 = i - i_1 = i - \frac{jpMi}{R_3 + jpL_2}$$

Substituting for i_1 and i_2 in (iii)

$$\left(i - \frac{jpMi}{R_3 + jpL_2}\right)R_1 - \frac{jpMi}{R_3 + jpL_2}R_2 - \frac{jpMi}{(R_3 + jpL_2)jCp} = 0$$

$$\text{or} \quad R_1 - \frac{jpM}{R_3 + jpL_2}\left[R_1 + R_2 + \frac{1}{jCp}\right] = 0$$

or $\quad R_1R_3+jpL_2R_1=jpM\,(R_1+R_2)+\dfrac{M}{C}$

Equating real and imaginary parts

$$R_1R_3=\frac{M}{C} \text{ and } R_1L_2=M\,(R_1+R_2)$$

$$\text{Hence} \qquad M = CR_1R_3$$

$$\text{and} \qquad L_2 = M.\frac{R_1+R_2}{R_1}$$

The current-leads from the mutual inductance coils should be properly connected to obtain the direction of currents as considered. R_3 includes the resistance of coil L_2 which should be known and added to the variable resistance in series with L_2.

Robinson's bridge for measurement of A.C. frequency : The arrangement is as shown in the circuit diagram (Fig. 7·28). The bridge is adjusted for null under a condition in which the two capacitances are equal and the resistances in the second and fourth arms are in the ratio 1 : 2 *i.e.* $R_4/R_2=2$.

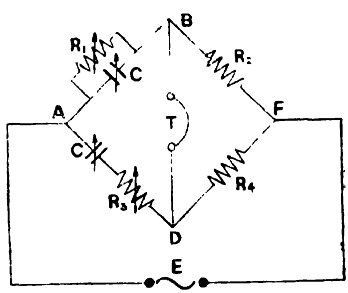


Fig. 7·28

For the balanced bridge, applying the null condition in the wheatstone bridge,

$$\frac{R_3+1/jCp}{Z_1}=\frac{R_4}{R_2}=2$$

$$\text{where} \qquad \frac{1}{Z_1}=\frac{1}{R_1}+jCp$$

$$\text{Hence} \quad (R_3+1/jCp)\left(\frac{1}{R_1}+jCp\right)=2$$

$$\text{or} \quad \frac{R_3}{R_1}+\frac{C}{C}+\frac{1}{jCpR_1}+jCpR_3=2$$

Equating real and imaginary parts,

$$\frac{R_3}{R_1}+1=2 \quad \text{or} \quad \frac{R_3}{R_1}=1$$

$$\text{and} \qquad \frac{1}{jCpR_1}+jCpR_3=0$$

$$\text{or} \qquad p^2=\frac{1}{C^2R_1R_3}$$

$$\text{If} \quad R_3=R_1=R, \qquad p^2=\frac{1}{C^2R^2}$$

$$\text{or} \quad 2\pi f=\frac{1}{CR}, \quad \text{So} \quad f=\frac{1}{2\pi CR}.$$

VII-12. TRANSFORMER

Principle : In the transformer a magnetic circuit is linked with two sets of windings, called the *primary* and the *secondary.* The elements of the scheme are shown in Fig. 7·29. There are two types of windings *viz.*, shell type and core type. When one of the windings is connected to an A.C. supply, an alternating magnetic flux is set up in the iron core. The flux is linked up with secondary and hence an alternating *emf* is induced in it. In principle the operation is a case of mutual induction.

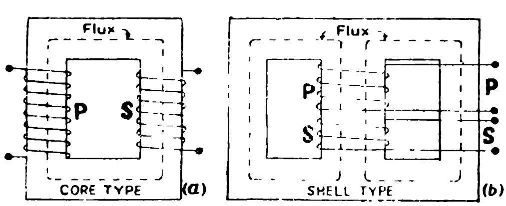


Fig. 7·29

Action of mutually inductive circuits : In the arrangement shown in Fig. 7·29 the entire flux developed in the core is embraced both by the primary and the secondary windings. Each winding has its own resistance and inductance. The applied *emf* in the primary has to overcome (i) the impedance drop in it, (ii) the induced *emf* in it due to the mutual induction caused by variations in the secondary. On the other

hand, the induced *emf* in the secondary is resisted only by the impedance drop in it.

Let the inductance, resistance and total number of turns of windings in the primary and the secondary respectively be L_1, R_1, n_1 and L_2, R_2, n_2. Let M be mutual inductance of the two coils.

When a current i_1 flows through the primary the flux linked with it is $N_1 = L_1 i_1$ and the flux linked with the secondary is $N_2 = M i_1$. Hence $L_1/M = N_1/N_2$.

If H be the field inside the core, $N_1 = \mu A H n_1$ and $N_2 = \mu A H n_2$, where A is the area enclosed by each turn of the windings and μ the permeability of the medium comprising the core.

$$\text{so} \quad \frac{L_1}{M} = \frac{N_1}{N_2} = \frac{\mu A H n_1}{\mu A H n_2} = \frac{n_1}{n_2}$$

$$\text{Hence} \quad M = L_1 \frac{n_2}{n_1} = L_2 \frac{n_1}{n_2} \quad \text{or} \quad \frac{M}{L_2} = \frac{n_1}{n_2}$$

Let an alternating *emf* $E_0 cospt$ be applied to the primary. The induced *emf* leads the current by $\pi/2$ in phase, so the primary and the secondary *emfs* may be expressed as

$$\text{Primary } emf \quad \mathbf{E}_1 = \mathbf{i}_1(R + jpL_1) + \mathbf{i}_2 jpM \qquad \cdots \qquad \text{(i)}$$

$$\text{Secondary } emf \quad \mathbf{E}_2 = -jpM\mathbf{i}_1 = \mathbf{i}_2(R + jpL_2) \qquad \cdots \qquad \text{(ii)}$$

$$\text{or} \quad \mathbf{i}_2 = \frac{-jpM\mathbf{i}_1}{R_2 + jpL_2} \qquad \cdots \qquad \cdots \qquad \text{(iii)}$$

$$\text{Hence} \quad \frac{\mathbf{i}_2}{\mathbf{i}_1} = -\frac{jpM}{R_2 + jpL_2}$$

Considering only magnitudes

$$\frac{i_2}{i_1} = \frac{Mp}{\sqrt{R_2^2 + p^2 L^2_2}}$$

$$\text{If } R_2 \to 0, \quad \frac{i_2}{i_1} = \frac{Mp}{L_2 p} = \frac{M}{L_2} = \frac{n_1}{n_2}$$

In an ideal transformer, there is no loss of energy and hence the rate of working should be the same in the two

circuits. So the power applied at the primary is equal to the power delivered at the secondary, hence

$$E_1 i_1 = E_2 i_2$$

$$\text{or} \quad \frac{E_1}{E_2} = \frac{i_2}{i_1} = \frac{n_1}{n_2}$$

The *emfs* in the two circuits are proportional to the respective number of turns. This is the *transformation ratio.* If the secondary turns are greater in number the *emf* produced at the secondary is higher than the primary applied *emf* in the same ratio as the number of turns. Such contrivance for obtaining an amplified voltage is known as a **step-up transformer**. On the other hand, if the secondary turns are less in number than the primary turns, the transformer is a **step-down transformer** converting a high voltage into a lower one.

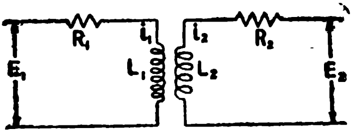


Fig. 7·30

REFLECTED IMPEDANCE : The secondary *emf* causes a change in the primary impedance as is shown below. Substituting the value of i_2 from equation (iii) in equation (i)

$$\mathbf{E}_1 = \mathbf{i}_1(R + jpL_1) + jpM\left(\frac{-jpM}{R_2 + jpL_2}\right)\mathbf{i}_1$$

$$\text{or} \quad \mathbf{E}_1 = \mathbf{i}_1\left[R_1 + jpL_1 + \frac{M^2p^2(R_2 - jpL_2)}{R_2{}^2 + p^2L_2{}^2}\right]$$

$$\text{or} \quad \mathbf{E}_1 = \mathbf{i}_1\left[\left(R_1 + R_2 \cdot \frac{M^2p^2}{R^2{}_2 + p^2L_2{}^2}\right)\right.$$

$$\left. + jp\left(L_1 - L_2 \cdot \frac{M^2p^2}{R^2{}_2 + p^2L_2{}^2}\right)\right]$$

$$\text{or} \quad \mathbf{E}_1 = \mathbf{i}_1[R_0 + jpL_0]$$

Considering the magnitudes only,

$$i_1 = \frac{E_1}{\sqrt{R_0{}^2 + p^2L_0{}^2}}$$

In the absence of the secondary, the primary current would have been

$$i_1 = \frac{E_1}{\sqrt{R_1^2 + p^2 L_1^2}}$$

So the effect of the secondary is to increase the primary resistance and to bring down a fall of primary inductance by amounts shown in the following equations,

$$R_o = R_1 + \frac{M^2 p^2}{R_2^2 + p^2 L_2^2} . R_2$$

$$\text{and} \quad L_0 = L_1 - \frac{M^2 p^2}{R_2^2 + p^2 L_2^2} . L_2$$

Effect of introducing a condenser in the secondary: If a capacitor C is put in series with the secondary coil, the total impedance drop in it becomes

$$\mathbf{E}_2 = -j\mathbf{i}_1 Mp = \mathbf{i}_2 \left[R_2 + j\left(pL_2 - \frac{1}{pC} \right) \right]$$

$$\text{or} \quad \mathbf{i}_2 = \frac{-jMp}{R_2 + j\left(pL_2 - \frac{1}{pC} \right)} \mathbf{i}_1$$

Substituting this value of $\mathbf{i}_2$ in (i)

$$\mathbf{E}_1 = \mathbf{i}_1 \left[R_1 + jpL_1 + \frac{M^2 p^2}{R_2 + j\left(pL_2 - \frac{1}{pC} \right)} \right]$$

$$\text{or} \quad \mathbf{E}_1 = \mathbf{i}_1 \left[R_1 + jpL_1 + \frac{M^2 p^2 \{ R_2 - j(pL_2 - \frac{1}{pC}) \}}{R_2^2 + (pL_2 - \frac{1}{pC})^2} \right]$$

$$\text{or} \quad \mathbf{E}_1 = \mathbf{i}_1 \left[R_1 + \frac{M^2 p^2 R_2}{R_2^2 + (pL_2 - \frac{1}{Cp})^2} + jp \left\{ L_1 - \frac{M^2 p (pL_2 - \frac{1}{pC})}{R_2^2 + (pL_2 - \frac{1}{pC})^2} \right\} \right]$$

$$\text{or} \quad \mathbf{E}_1 = \mathbf{i}_1 \left[R_0 + jpL_0 \right]$$

$$\text{where} \quad R_0 = R_1 + \frac{M^2 p^2 R_2}{R_2^2 + (pL_2 - \frac{1}{pC})^2}$$

$$\text{and} \quad L_0 = L_1 - \frac{M^2 p (pL_2 - \frac{1}{pC})}{R_2^2 + (pL_2 - \frac{1}{pC})^2}$$

The apparent decrease is inductance has been somewhat compensated for. As shown in the numerator of term the amount of dimunition being $M^2p(pL_2-\frac{1}{pC})$ in place of $M^2p^2L_2$ is somewhat a smaller quantity. It may noted that dimunition depends upon the square of frequency.

E.M.F. equation of a transformer : Let n_1 and n_2 be the total number of turns in the primary and the secondary windings respectively. Let N be the maximum flux linked with the windings and f the frequency of supply. Since the flux grows from zero to maximum in a time $t=1/4f$, so

$$\text{Average rate of flux change} = \frac{N}{t}=4fN$$

$$\text{Average } emf \text{ induced in each turn} = 4fN$$

Average total emf in n_2 turns of secondary is given by $E_2=4fNn_2$.

If each turn encloses an area A and if the field in the core of permeability μ be H, then $N=\mu AH$, hence

$$E_2 = 4\mu fAHn_2$$

In a sinusoidal emf, $\dfrac{R.M.S.\ value}{Average\ value} = 1{\cdot}11$ (form factor)

Hence $R.M.S.$ value of the induced emf is obtained as

$$E_{r.m.s.}=4{\cdot}44\mu fAHn_2 \ c.g.s. \text{ units}$$

In practical units $E_{r.m.s.}=4{\cdot}44\mu fAHn_2\times10^{-8}$ volts

If N is the total flux, $E_{r.m.s.}=4{\cdot}44fNn_2\times10^{-8}$ volts

Leakage and Losses : In practical forms of transformer the induced emf does not reach the theoretical value which is applicable for an ideal arrangement. For several reasons, a transformer cannot, in practice, be made to attain the ideal state of perfections.

MAGNETIC LEAKAGE : The windings of an ideal transformer are assumed to possess no impedance. This may be true only if the ohmic resistance becomes negligible and the reactance is zero. But in practical form of windings reactance arises due to self-induction. In an ideal transformer

the flux linked with the primary due to its own current is neutralised by the flux produced by the secondary. Hence there is no flux linked with the primary exclusively due to its own current and hence there is no reactance arising out of self-inductance. But this ideal behaviour is not realised in practice. There is always some amount of flux which though linked with one set of windings avoids linkage with the other (Fig. 7·31). This is *magnetic leakage*. There are such fluxes for both primary and secondary. These fluxes, called *leakage flux*, produce in their respective windings *emfs* due to self-inductance and causes inductive reactance in the circuit. In effect these reactances are equivalent to the addition of an *inductive coil in series* with each of the windings. So a transformer with magnetic leakage is equivalent to an ideal transformer with extra inductive coils connected separately with both primary and secondary windings.

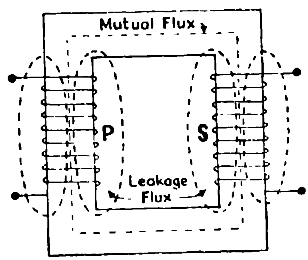


Fig. 7·31

COPPER LOSS : The ohmic resistance of the windings cause a loss of power. If R be the effective resistance of the windings the power loss caused by the heating effect due to a current i is Ri^2.

Again if R_p and R_s be the ohmic resistances of the primary and secondary windings respectively the reflected impedance causes the effective value of the resistance R_p to rise to R_1 given by

$$R_1 = R_p + \frac{M^2p^2}{R_2{}^2+p^2L_2{}^2} \cdot R_s$$

Thus R_1 depends somewhat on the frequeney of the *emf*.

The accompanying diagram (Fig. 7·32) shows an equivalent circuit in which the resistance causing copper loss and leakage reactance have been taken out of the respective windings. The primary and the secondary impedances are given by

$$Z_1 = \sqrt{R_1{}^2+X_1{}^2} \text{ and } Z_2 = \sqrt{R_2{}^2+X_2{}^2}$$

This impedance causes some drop of voltage in the respective windings. If V_1 is the input voltage at the primary the

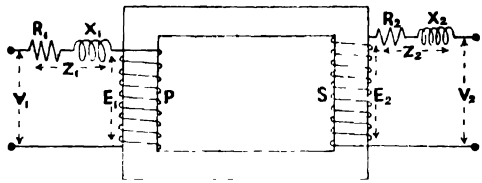


Fig 7·32

voltage E_1 effective for transformation is given by

$$\mathbf{E}_1=\mathbf{V}_1-\mathbf{i}_1(R_1+jX_1)=\mathbf{V}-\mathbf{i}_1 Z_1$$

Similarly for the secondary, the available effective voltage V_2 is related to the output voltage E_2 produced by transformation as

$$\mathbf{V}_2=\mathbf{E}_2-\mathbf{i}_2(R_2+\mathrm{j}X_2)=\mathbf{E}_2-\mathbf{i}_2\mathbf{Z}_2$$

The transformation ralio $(n_1 : n_2)$ is applicable to $(E_2 : E_1)$ and not to $(V_2 : V_1)$.

IMPEDANCE REFERRED TO ONE WINDING: If two winding resistances are R_1 and R_2, the total number of turns n_1 and n_2 and current, i_1 and i_2 respectively, then we have

$$i_2=\frac{n_1}{n_2}.i_1 \text{ and so}$$

$$\text{Total copper loss}=R_1 i_1{}^2+R_2 i_2{}^2$$

$$=\left[R_1+R_2\left(\frac{n_1}{n_2}\right)^2\right] i_1{}^2$$

Let us imagine that that the secondary resistance is zero and to compensate for this the primary resistance is increased to a value indicated by R_{T-1}. Then for the same loss we shall have

$$R_{T-1}=R_1+R_2\left(\frac{n_1}{n_2}\right)^2$$

R_{T-1} is called the *total resistance referred to the primary.*

This may be obtained from a different consideration in general for a resistance or reactance. Regarding the transference of resistance or reactance, we may consider that the secondary resistance referred to the primary is that resistance which produced the same percentage of potential drop as actually takes place in the secondary.

Consider a transformer with transformation ratio $n_1 : n_2$. A secondary current i_2 produces a resistance drop $R_2 i_2$ in a resistance R_2. A potential drop $R_2.i_2.\frac{n_1}{n_2}$ in the primary would by transformation cause the same drop $R_2 i_2$ in the secondary. Since $i_2 = i_1 \frac{n_1}{n_2}$, so $R_2.i_2.\frac{n_1}{n_2} = R_2 i_1 \left(\frac{n_1}{n_2}\right)^2$. Therefore the value of secondary resistance R_2 referred to the primary is $R_2 \left(\frac{n_1}{n_1}\right)^2$. That is for total resistance referred to the primary

$$R_{T-1} = R_1 + R_2 \left(\frac{n_1}{n_2}\right)^2$$

Similarly, $$R_{T-2} = R_2 + R_1 \left(\frac{n_2}{n_1}\right)^2,$$

which stands for total resistance referred to the secondary.

Again considering the whole of reactance as being transferred to either primary or secondary, we have

$$X_{T-1} = X_1 + X_2 \left(\frac{n_1}{n_2}\right)^2, \text{ and } X_{T-2} = X_2 + X_1 \left(\frac{n_2}{n_1}\right)^2$$

Hence for total impedance similarly referred

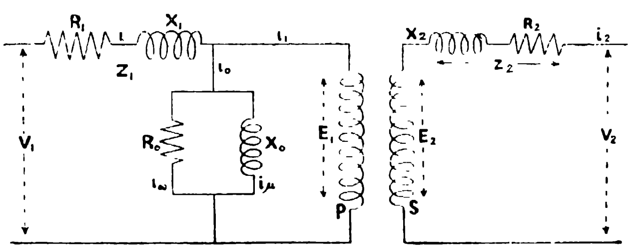
$$Z_{T-1} = Z_1 + Z_2 \left(\frac{n_1}{n_2}\right)^2 \text{ and } Z_{T-2} = Z_2 + Z_1 \left(\frac{n_2}{n_1}\right)^2$$

IRON CORE LOSSES : These are losses due to hysteresis and eddy current in the core produced by alternating flux. Since the core flux is constant practically at all loads, iron losses may be regarded to be independent of load. Laminating the core decreases eddy current and use of materials having low hysteresis and high electrical resistance, such as silicon

steel reduces both the losses. In an equivalent circuit these two are simulated by a pure inductance (X_o) and a non-inductive resistance (R_o) connected in parallel across the primary. The primary input of the transformer at no load (open secondary circuit) measures the core loss. Let i_o be the no-load current in the primary and $cos\phi$ be the power factor. Then i_w and i_μ, the working and the magnetising components respectively are given by $i_w = i_o cos\phi$ and $i_\mu = i_o sin\phi$. If E_1 be the primary *emf* and P the power consumed then

$$cos\phi = \frac{P}{E_1 i_o} \text{ and } X_o = \frac{E_1}{i_\mu} \text{ and } R_o = \frac{E_1}{i_w} .$$

EQUIVALENT CIRCUIT : In such a circuit for an imperfect transformer, the resistance and the leakage reactance are taken as external to the windings in series both with the



(*a*) Fig. 7 33 (*b*)

primary and the secondary windings. The hysteresis and eddy current losses consuming the no-load current is represented as current consumed in a pure inductance and a non-inductive resistance joined in parallel with the respective windings. The value of the effective voltage E_1 at the primary windings (taken as ideal) serving solely to change the value of the *emf* according to turns ratio is obtained as the vectorial difference of the applied *emf* V_1 and voltage drop iZ_1.

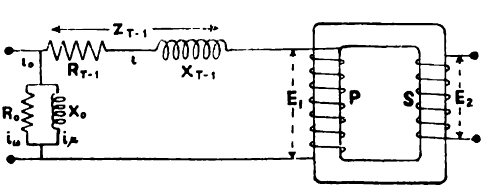


Fig. 7·34

Referring to the total impedance referred to the primary we shall have an equivalent circuit as shown in Fig. 7·34. The *emf* V_2 available at the ends of the secondary terminals is given by

$$\mathbf{V}_2 = \frac{n_2}{n_1}\left[\mathbf{V}_1 - \mathbf{i}_1 \mathbf{Z}_{T-1}\right]$$

The values of R_{T-1} and X_{T-1} may be directly obtained by short-circuiting the secondary windings. In such a case $V_2=0$; since the whole of the impedance has been taken outside the windings being referred to the primary, there cannot be any internal drop of voltage. Hence if V_s be the primary voltage (applied to a produce a current i_s) the whole of it is utilised in overcoming the total impedance Z_{T-1}. So

$$Z_{T-1} = \frac{V_s}{i_s}$$

If the power is measured as P_s, this will be due to copper loss. hence

$$R_{T-1} = \frac{P_s}{i_s^2}$$

$$\text{and} \quad X_{T-1} = \sqrt{Z_{T-1}^2 - R_{T-1}^2}.$$

Efficiency of a Transformer: The efficiency is measured in different ways. The *ordinary* or the *commercial* efficiency is defined as the ratio of the power output to power in-take in watts.

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}} = \frac{\text{Output}}{\text{Output} + \text{iron and copper losses.}}$$

$$\text{Efficiency} = \frac{\text{Input} - \text{losses}}{\text{Input}} = 1 - \frac{\text{losses}}{\text{Input}}$$

$$\text{or Efficiency } \eta = 1 - \frac{\text{losses}}{\text{Output} + \text{losses}}$$

Since efficiency is the ratio of the actual power output to power input, it depends upon the power factor as well as the load.

All-day efficiency is measured on energy basis and is expressed as

$$\text{All-day efficiency} = \frac{\text{Output in } K\text{-}W \text{ hours}}{\text{Input in } K\text{-}W \text{ hours}}$$

It is less than commercial efficiency. The reason lies in the fact that core losses go on for the whole day even when there is no load. A transformer connected to the line for the whole day work indicates less efficiency if the load is not continuous.

CONDITION FOR MAXIMUM EFFICIENCY : Let the primary *emf*, current and resistance be E_1, i_1 and R_1 and the similar components for the secondary be E_2, i_2, R_2. The power factor is $\cos\phi$.

$$\text{Input} = E_1 i_1 \cos\phi$$

$$\text{Primary copper loss} = R_1 i_1^2$$

$$\text{Secondary copper loss} = R_2 i_2^2$$

$$\text{Total copper loss} = \text{Vector sum of } R_1 i_1^2 + R_2 i_2^2$$

$$= R_{T-1} i_1^2 \, .$$

R_{T-1} be such as to represent the equivalent primary resistance producing an equal amount of heating.

$$\text{Iron loss} = W \text{ (constant)}$$

$$\text{Efficiency } \eta = \frac{\text{Input}-\text{losses}}{\text{Input}} = 1 - \frac{\text{losses}}{\text{Input}}$$

$$\text{or} \quad \eta = 1 - \frac{i_1^2 R_{T-1}}{E_1 i_1 \cos\phi} - \frac{W}{E_1 i_1 \cos\phi}.$$

For η to be maximum $\frac{d\eta}{di_1} = 0$

$$\text{So} \quad -\frac{R_{T-1}}{E_1 \cos\phi} + \frac{W}{i_1^2 E_1 \cos\phi} = 0$$

$$\text{or} \quad R_{T-1} = \frac{W}{i_1^2}$$

$$\text{or} \quad R_{T-1} . i_1^2 = W$$

That is, copper loss = iron loss

Thus the efficiency is maximum when the copper loss is equal to the iron loss.

Auto-transformer : This type of transformer has one set of winding only, a part of this forming the secondary is

common to primary as well. Its operation is similar to that of a two-winding transformer. Since it uses less copper it is cheaper but its uses are limited. *AB* forms the primary winding having n_1 turns and *AC* is taken as the secondary winding taking n_2 turns $(n_1 > n_2)$. The *emfs* in the two sets of the terminals may be obtained by considering *AB* as a choke coil with closed magnetic circuit.

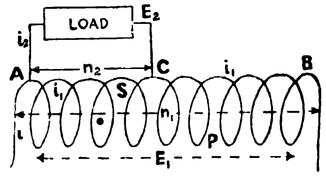


Fig. 7·35

Since each turn is linked by the same flux the voltage drop per turn will be constant and therefore if a tapping is taken at some intermediate point *C*, the voltage between *A* and *C* will be $V_2 = \frac{n_2}{n_1} V_1$, where V_1 is the *emf* applied between *A-B*. In the case $V_2 < V_1$ But if the windings are used in the reverse way, the primary turns being less than the secondary turns, $V_2 > V_1$. Thus the appliance can be used as a step-down or a step-up transformer. The supply current is divided in two branches as i_1 and i_2, i_2 being the vector difference of i and i_1.

In practice an auto-transformer is used as a booster in a distribution cable to correct for the voltage drop. Its main application is to supply a reduced voltage for the starting of induction motors and synchronous motors.

ILLUSTRATIVE EXAMPLES

1. In a transformer the primary and the secondary windings consist of 80 and 600 turns respectively. What will be the secondary voltage if the primary is connected to a 240-volt supply? If the power of transformation is 10 KVA, calculate the primary and the secondary currents. (Neglect all losses).

Solution : $\frac{E_1}{E_2} = \frac{n_1}{n_2}$

$$\frac{E_2}{240} = \frac{600}{80} \text{ or } E_2 = \frac{600 \times 240}{80} = 1800 \text{ volts.}$$

$$E_1 i_1 = 10 \times 10^8 \text{ or } i_1 = \frac{10000}{240} = 41{\cdot}6 \text{ amperes.}$$

$$E_2 i_2 = 10 \times 10^8 \text{ or } i_2 = \frac{10000}{1800} = 5{\cdot}5 \text{ amperes.}$$

2. *Calculate the maximum flux produced inside the core of a transformer having 1500 turns in the secondary, when a 66000-volt-50 cycle emf is obtained at the output. (Neglect all losses)*

Solution : $E_2 = 4{\cdot}44 n_2 N f \times 10^{-8}$ volts.

$$N = \frac{E_2 \times 10^8}{4{\cdot}44 n_2 f} = \frac{66000 \times 10^8}{4{\cdot}44 \times 1500 \times 50} = 12{\cdot}3 \times 10^8 \text{ maxwells.}$$

8. *In a 50 KW transformer the iron loss is 500 watts and the full load copper loss is 600 watts. Calculate the efficiency at half-load at 0·9 power factor.*

Solution : At half-load copper loss is

$$\frac{1}{4} \times 600 = 150 \text{ watts} = 0{\cdot}15 \ KW.$$

Iron loss remains constant and is 500 watts$=0{\cdot}5$ *KW.*

$$\text{At } 0{\cdot}9 \text{ power-factor output } = \frac{1}{2} \times 50 \times 0{\cdot}9 = 22{\cdot}5 \ K.W.$$

$$\text{Efficiency} = 1 - \frac{losses}{output + losses} = 1 - \frac{0{\cdot}15 + 0{\cdot}5}{22{\cdot}5 + 0{\cdot}15 + 0{\cdot}5}$$

$$= 1 - \frac{0{\cdot}65}{45{\cdot}65} = 1 - 0{\cdot}028 = 0{\cdot}972$$

Efficiency is 97·2 percent.

VII-13. HIGH FREQUENCY CURRENT FLOW

Skin effect : When a steady current flows through a conductor of uniform cross-section, the current density is uniform inside the conductor. But when an alternating current is applied at the ends of a conductor there is concentration of current in outer layers. If the frequency is very high, the flow is almost confined to the surface layer. This is known as *Skin effect.*

This behaviour of a conductor regarding the flow of current leads to the conclusion that for some reason or other the

impedance of the inner core becomes greater than that of the outer shell. It may be realised if the outer shell of a conducting wire is regarded as a conductor separate from the central core, the two being in parallel combination. In such a case an outer source of *emf* will perfer to send more current by that path which has a lower value of impedance comprised of reactance and resistance.

Let a conductor of circular cross-section be divided into concentric shells at right angles to its length. Let us consider two such elements of equal cross-sectional area, one a central (A) and the other an outer cylindrical shell (B) as shown in Fig. 7·36. The central core has a greater self-inductance than the outer shell for reason discussed below.

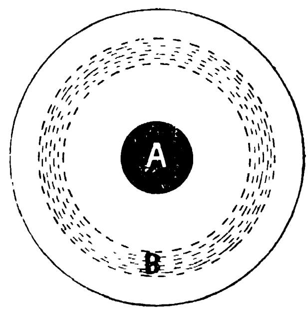


Fig. 7·36

The total magnetic flux due to a current flowing through B is less than the flux due to the same current flowing through A, by the amount that fills the space between A and B. This leads to the fact that the self-inductance (L_A) of it is greater than that of B (L_B). It may also be realised by considering that self-induction of a conductor really measures the energy of the field *i.e.* the work done in establishing the flux associated with it when unit current flows through it. The field for a linear current i obtained as $H=2i/r$ is greater in regions near the centre from that away from it. Hence we may conclude that $L_A>L_B$. Further, the mutual inductance (M) of the two is equal to L_B since the flux linked with B is also linked with A, but all of the flux linked with A is not linked with B. So since $L_A>L_B$ and $M=L_B$, $L_A>M$.

Let the resistance of each element for steady current be R. When an alternating *emf* is applied to both regarded as two separate conductors in parallel, each circuit may be considered to have the other as its secondary and as such resistance and

inductance of both are altered (see reflected impedance §VII-12) as shown below.

$$R_A = R + \frac{M^2p^2}{M^2p^2+R^2}.R = R\left(1+\frac{M^2p^2}{M^2p^2+R^2}\right)$$

$$\text{and } R_B = R + \frac{M^2p^2}{L_A{}^2p^2+R^2}.R = R\left(1+\frac{M^2p^2}{L_A{}^2p^2+R^2}\right)$$

$$\text{Again } L_a = L_A - \frac{M^2p^2}{M^2p^2+R^2}.M$$

$$\text{and } L_b = M - \frac{M^2p^2}{L_A{}^2p^2+R^2}.L_A$$

The resistance of each is increased due to the effect of the other but the increase in A is greater, *i.e.* $R_A > R_B$ since $L_A > M$. If p is very high, R_A becomes appreciably greater in comparison with R_B.

The changes in reactance of the inner core and outer shell may also be considered. For a material of good conductivity R is small and if p is high, R^2 becomes negligible in presence of M^2p^2 and $L_A{}^2p^2$. In such a case.

$$L_a = L_A - M \text{ and } L_b = M - \frac{M^2}{L_A} = \frac{M}{L_A}(L_A - M)$$

$$\text{Since } \frac{M}{L_A} < 1, \text{ hence } L_a > L_b$$

Hence if p is very high the reactance of the inner core becomes much greater than that of the outer shell. This becomes more prominent when the material has a high permeability whereby L_A is largely increased, M remaining practically unchanged. This again causes the crowding of current into outer layers. Thus for good conductors the reactance becomes the main factor in controlling current and so at high frequencies the current confines itself mainly to the outer surface.

For this effect conductors meant to carry high frequency alternating current are prepared of a number of stranded fine wires insulated from one another. This provides for a large surface area for any given area of cross-section. It may be

mentioned that high frequency current may be passed through human body without injury, the current flowing mainly over the skin.

Skin effect otherwise means *apparent rise in resistance* of a wire for a rapidly alternating current. According to Rayleigh the effective resistance R' of a wire of radius a, specific resistance ρ for a high frequency f having a steady resistance R is given by

$$R' = \frac{R\sqrt{\pi^2 f \mu a^2}}{\rho}.$$

Tesla coil : When a condenser is discharged through an inductance of low resistance, the discharge becomes oscillatory, the frequency f, the inductance L and the capacitance C being related as $f = 1/2\pi\sqrt{LC}$.

In the arrangement shown (Fig. 7·37) the secondary (S_1) of a step-up transformer is connected with a capacitance C of low value (but capable of withstanding high voltage pressure) and a coil (P_2) of small inductance L_2 through a spark gap ($S.G$). The coil P_2 again forms the primary of an air core transformer of which the secondary S_2 is made of a large number of turns. One end of this secondary is earthed and the other is connected a sparking knob K.

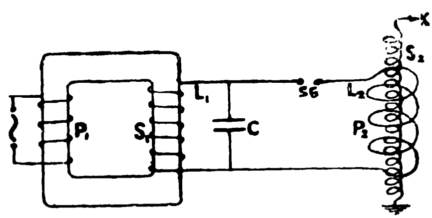


Fig. 7·37

When an A.C. voltage is applied at P_1 the secondary S_1 excites the circuit P_2C initiated by a discharge at the spark gap of the charge accumulated in C. Since both L and C are of low value a very high frequency current is set up. This again produces a high voltage in secondary S_2 attached to it. This high frequency high voltage discharges at the knob. The discharge through air may excite a fluorescent lamp and may be safely pased through the human body with no injury showing the action of the skin effect.

VII-14. ELIHU-THOMSON EFFECT

Repulsion between an A.C. electromagnet and a conductor : Let us consider a long coil (A) with an iron core (Fig. 7·38) carrying a current $i_0 \sin pt$. A closed ring (B) of light metal is placed inside the cylindrical core projecting beyond the coil. Let L be the self-inductance of B and R its resistance. M is the mutual inductance of A-B. The induced *emf* in B regarded as secondary of A is

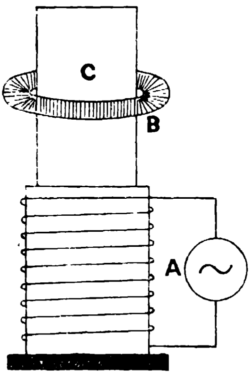


Fig. 7·38

$$e = -M\frac{di}{dt} = -M\frac{d}{dt}(i_0 \sin pt)$$

or $$e = -Mpi_0 \cos pt = Mpi_0 \sin\left(pt - \frac{\pi}{2}\right)$$

The induced current in B is

$$i = \frac{Mpi_0}{\sqrt{R^2+L^2p^2}} \sin\left(pt - \frac{\pi}{2} - \theta\right),$$

where $\theta = \tan^{-1}\frac{Lp}{R}$

Force between A and B is proportional to the product of current in A and B. Hence,

$$\text{Force } F \propto \frac{i_0 \sin pt \; Mpi_0 \sin\left[pt - \left(\frac{\pi}{2}+\theta\right)\right]}{\sqrt{R^2+L^2p^2}}$$

$$\text{or } F \propto \frac{i_0^2 Mp[\sin^2 pt \cos(\frac{\pi}{2}+\theta) - \frac{1}{2}\sin 2pt \sin(\frac{\pi}{2}+\theta)}{Z}$$

where Z stands $\sqrt{R^2+L^2p^2}$

Considering the mean value, $F \propto \dfrac{-i_0^2 Mp \sin\theta}{Z}$

The negative sign in the expression implies that the force is a repulsive one. If may also be evident by considering the ring acting as secondary of step-down transformer, the solenoid being the primary. The fields due to the two are always opposed to each other, like poles being produced at the end of the solenoid and the face of the ring which are adjacent to each other. Hence a repulsive force is produced which throws the ring violently in the upward direction. If the ring is prevented from jumping it gets heated.

EXERCISES ON CHAPTER VII

7·1. Explain the terms Mean value, Root mean square value and Peak value of an alternating current. Find the relations between them.

A coil of 50 turns each of area 200 *sq.cm* is rotated at 1200 *r.p.m.* in a uniform field of flux density of 5000 maxwells per *sq.cm.* Calculate (*a*) the peak and the effective values of the *emf* generated (*b*) the frequency and (*c*) the instantaneous *emf* 0·005 sec, after it has passed through the maximum value.

[*Ans* : (a) 125·6 volts, 44·4 volts (b) 20 c.p s. (c) 101·6 volts]

7·2. Define the form factor of an alternating wave. Calculate its value for a purely sinusoidal current.

7·3. Obtain an expression for the amplitude of the current in series circuit containing a resistance and a self-inductance when an *emf* $E_0 \sin pt$ is applied. Explain what is meant by impedance.

An *emf* of 200 volts (*r.m.s.*) at 50 cycles per second is applied to a circuit consisting of a resistance 50 ohms in series with an inductance of 100 *mH*. Calculate the impedance, the current and the phase angle.

[*Ans* : 59·65 ohms, 3·35 amp, 32°·1]

7·4. Draw vector diagrams to represent current and voltage in A.C. circuits containing (a) a resistance and an inductance (b) a resistance and a capacitance. Hence obtain the current in a general series circuit consisting of resistance, an inductance and a capacitance in series to which a sinusoidal *emf* is applied.

Define reactance and impedance of such a circuit. Explain the terms admittance and susceptance.

7·5. Write down the differential equation for the *emf* of an A.C. circuit containing an inductance L, a resistance R and a capacitance C in series to which an *emf* $E_0 \sin pt$ is applied. Solve the equation to obtain the instantaneous current in the circuit.

Calculate the impedance at 50 cycles per second of a series circuit containing a resistance 5 ohms, an inductance 0·05 henry and a capacitance 300 micro-farad. Obtain the phase angle of the current.

[*Ans* : 7·14 ohms, $\phi=45^{\circ}\cdot6$ lagging]

7·6. Explain the term power factor of an A.C. circuit. What is meant by wattless component of an alternating current ?

An *emf* of 200 volts (r.m.s.) at 50 c.p.s. is applied to an inductive coil of resistance 10 ohms and self-inductance 0·05 henry. Calculate the rate at which the energy is dissipated in the circuit. [*Ans* : 1079 watts.]

7·7. What is a choke coil ? Explain its utility.

Show that the power absorbed by a circuit carrying an alternating current and having an impedance is $Ei \cos\phi$, where E and i are the virtual values of *emf* and current respectively and ϕ is the phase difference between them.

An electric glow lamp runs at 100 volts D.C. taking 0·5 amp. current. It is connected to 100 volts (r.m.s.) A.C. supply at 50 c.p.s. and the current is found to be 0·34 amperes (r.m.s.). Calculate the inductance of the filament of the lamp.

[*Ans* : 0·66 henry]

7·8. Discuss the distinction between resistance and reactance of an electric circuit.

An alternating *emf* of 200 volts (r.m.s.) at 50 c.p.s. is applied to a series circuit containing an inductance of 5 henries and resistance of 1000 ohms in series. Calculate the r.m.s. value of the current flowing on the circuit and its phase in relation to the voltage. What is the power consumed ?

[*Ans* : 0·107 amp., $\phi=57^{\circ}\cdot5$, 5·68 watts]

7·9. Describe and explain the working principle of an Earth inductor. State its uses and describe the methods involved.

7·10. A circuit consists of three branches, the impedances of which at 50 c.p.s. are respectively

$$Z_1=12+3j,\ Z_2=5-4j,\ Z_3=7+8j$$

Determine the values and phases of the current when the three branches are in (*a*) parallel (*b*) series in a 200-volt-50 cycle supply.

[*Ans* : (*a*) 52·4 amp., 15°·2 (leading),
(*b*) 7·746 amp., 15°·2 (lagging)]

7·11. What is electrical series resonance ? Discuss voltage and current resonances in such a circuit and indicate the conditions to obtain these.

A circuit containg an inductive resistance of 0·04 henry and 4 ohms is in series with a capacitor of 25 μF. Find the resonant frequency and current in the circuit at this frequency for an *emf* of 200 volts in the circuit.

[*Ans* : 159 c.p.s., 25 amp.]

7·12. What are acceptor and rejector circuits ? Explain their action. 'A rejector circuit at resonance shows current magnifiication whereas the acceptor circuit at resonance causes voltage magnification'—Discuss this statement fully.

7·13. Obtain the condition for sharp resonance in a series circuit. What is 'pass-band' of a resonant circuit ?

7·14. Describe a parallel resonant circuit. Such a circuit may be used to filter out the current of a particular frequency—Explain.

What is dynamic resistance ? What is *Q*-factor ?

7·15. At what frequency should a current in a *L*-*R*-*C* series circuit be independent of capacitance and inductance ? Discuss its importance.

Calculate the voltage magnification at the terminals of an inductance of 0·8 henry and resistance 10 ohms in series with a capacitor of $5\mu F$ when an *emf* of 100 volts is applied at the resonant frequency.

[*Ans* : 20·1]

7·16. What is a transformer ? Show that the secondary of a transformer produces an apparent increase in resistance and decrease in inductance of the primary. How can this be minimised ?

7·17. Work out the theory of a transformer and explain its function. Investigate the effect of secondary on the impedance of the primary.

7·18. The maximum flux-density through the core of a step-down transformer is 10^6 maxwells. If the input is 220 volts at 50 c.p.s. and the transformation ratio is 1 : 5, find the number of primary turnings neglecting the resistance of the windings.

[*Ans* : 500]

7·19. Obtain the *emf* equation of a transformer. Discuss the leakage and losses in relation to an imperfect transformer. Draw an equivalent circuit for such a transformer.

What is meant by efficiency of transformer ? Obtain the condition for maximum efficiency.

7·20. Explain the apparent rise in resistance of a wire with frequency for a rapidly alternating current flowing through it. Discuss its practical consequences.

7·21. Obtain the condition for a balance in the generalised wheatstone bridge having impedance in each branch. Apply it in Maxwell's bridge meant for measurement of self-inductance.

7·22. Describe Anderson's method of measuring self-inductance.

7·23. Explain how by means of wheatstone bridge applied to A.C. circuits the frequency of a sinusoidal *emf* may be determined.

7·24. Describe an accurate method of determination of small capacitance relative to a standard capacitor.

7·25. Write notes on : (a) Skin effect (b) Tesla coil (c) Elihu-Thomson effect (d) Auto-transformer.

7·26. Explain the term negative resistance of a circuit. Deduce the conditions that a circuit may be oscillatory.

CHAPTER VIII

ELECTRICAL TECHNOLOGY

VIII-1. BASIC PRINCIPLES OF A GENERATOR

E.M.F. generated in a straight conductor : Consider that a straight conductor O of length l centimeters is moving with a uniform velocity of v centimetres per second in a uniform magnetic field of flux density B, in a direction at right angles to the lines of induction and perpendicular to its length (Fig. 8·1)

Let the conductor move through a distance $OP=d$ in t seconds, so that the flux cut by it is

$$N=B\times l\times d$$

The induced *emf*

$$e=-\frac{dN}{dt}=-\frac{Bld}{t}=-Blv.$$

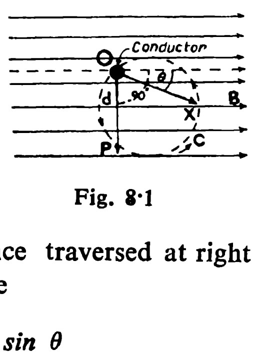


Fig. 8·1

More generally let the conductor move a distance $OX=x$ in a direction inclined at an angle θ with the lines of force, then the distance traversed at right angles to the field is given by $x \sin \theta$, hence

$$N=Blx \sin \theta \quad \text{and} \quad e=-\frac{dN}{dt}=-Blv \sin \theta$$

If the conductor moves parallel to the field $\theta=0$ then $e=0$, since $\sin \theta=0$. The *emf* is maximum when $\theta=90°$.

If B in the above equation be in maxwells, the *emf* generated in the conductor expressed in volts, will be $Blv \sin\theta\times 10^{-8}$ volts.

E.M.F. generated in a rectangular coil : Movement of a conductor in a magnetic field as described in the above may be continuous if the conductor instead of proceeding in a straight line rotates in the field in a circular path, as indicated by the curve drawn in broken line. Such a motion is realised in the rotation of a rectangular coil.

Let a rectangular loop *abcd* (Fig. 8·2) of a conducting material be rotated with uniform angular velocity ω about an axis XY perpendicular to the field. If $2r$ be the breadth (*ab* or *cd*) the linear velocity of the rotating loop is $v=\omega r$. During rotation the planes of rotation of *ab* and that of *cd* remain parallel to the field and as such they cut no lines of force and there is no induced *emf* in these portions. But the side *ad* or *bc*, each of length l, remains perpendicular to the field and their direction of motion with respect to the field denoted by θ, continually changes so that θ varies from 0 to 2π. These conductors cut the lines of force during rotation and the induced *emf* in *ad* and in *bc* are mutually opposite in direction. But the *emfs* in the two act cummulatively round the loop. At the instant when the conductors are moving in a direction θ with the field, the induced *emf* is given by

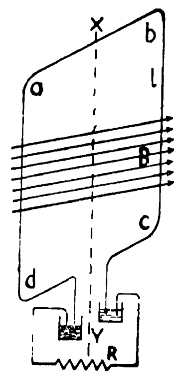


Fig. 8·2

$$e=2\ Blv\ sin\ \theta.$$

Since the side *ad* or *bc* while rotating always move in a direction normal to the plane of the coil, the angle θ may be regarded as the angle made by this norma! with the field at any instant.

If the loop is replaced by a coil of n turns, the total induced *emf* at any instant is given by

$$E=2n\ Blv\ sin\ \theta$$

If time be reckoned in such a way that when $t=0$, $\theta=0$, *i.e.* when the normal to the coil is parallel to the field then at any subsequent instant, $\theta=\omega t$, which gives the angular shift of the coil normal from its coincidence position relative to the field. Hence

$$E=2nBl\omega r\ sin\ \omega t=nB(2r.l)\omega.\ sin\ \omega t$$

or $E=An\omega B\ sin\ \omega t$ (*e.m.* units),

where A is the area of each loop of the coil.

or $E = E_0 \sin \omega t$, where $E_0 = An\omega B$

or $E = An\omega B \sin \omega t \times 10^{-8}$ volts.

It may be noted that AnB is the total flux through the coil when its normal is parallel to the field. The *emf* is maximum when the coil normal is perpendicular to field and the flux through the coil is minimum as in this position the rate of change of flux is maximum.

Principle of a Dynamo : Generator or Dynamo is a contrivance for converting mechanical energy into electrical energy. The working of a generator is based upon the induced *emf* due to electromagnetie induction in rotating coil as discussed in the foregoing section.

Suppose a rectangular coil of n turns each of area A rotates with a circular frequency, *i.e.* angular velocity p. Frequency giving the number of revolutions per second is obtained as $f = p/2\pi$. The rotation is about an axis perpendicular to a magnetic field of induction B. In air B is same as the magnetic force H and in a medium of permeability μ, $B = \mu H$. If the normal to the coil is initially (when $t = 0$) parallel to the field, at any subsequent instant t it will be displaced through an angle $\theta = pt$. Magnetic flux linked with the coil in this position is

$$N = An\mu H \cos pt$$

Induced *emf* $E = -\frac{dN}{dt} = An\mu Hp \sin pt = E_0 \sin pt$. The magnitude of the *emf* is sinusoidal, its amplitude is given by $E_0 = An\mu Hp$, which again is the instantaneous *emf* when the normal to the coil is perpendicular to the field. A current varying in a simple harmonic manner is obtained in such a rotating coil. Since the current is reversed in direction alternately during rotation it is called *Alternating current*.

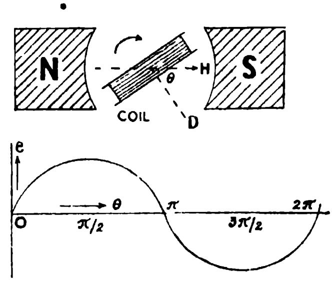


Fig. 8·3

In a dynamo there is arragement to transfer the current from the rotating coil to an

external circuit in which alternating or unidirectional (direct) current is obtained. The generators are therefore clasfified as D.C. and A.C. generators.

VIII-2. D. C. GENERATORS

Essential parts : The practical form of a D. C. generator consists of the following components : (*i*) a magnet to produce a field, (*ii*) an assemblage of conductors placed in the field, (*iii*) an arrangement for producing motion in the conductors, (*iv*) a contrivance for transfer of current to any external circuit. All these are accommodated in the *field-magnet* and the *armature*.

(1) FIELD MAGNET : It may have one or more pairs of opposite poles. The field system cosisting of four poles is shown in the diagram (Fig. 8·4-a). It consists of (i) a cylindrical yoke which serves the double purpose of carrying the magnetic flux and of acting as a frame of the machine, (ii) the coils carrying current to produce the field and (iii) the pole shoes.

(2) ARMATURE : It is a cylindrical structure made up of sheets or stampings of electrical sheet iron or steel. The armature core has a series of longitudinal slots for housing

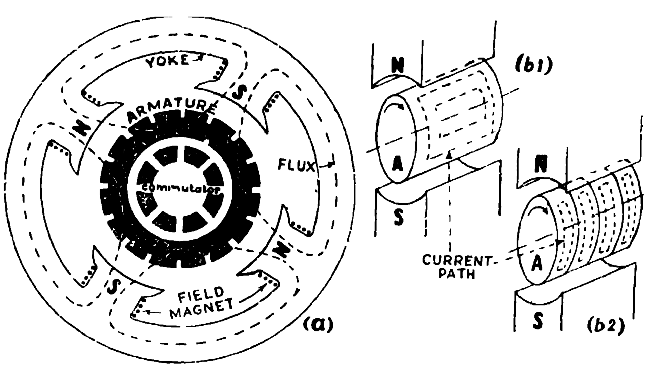


Fig. 8·4

the conductors forming the coil. The material of the core increases the flux and to reduce the losses due to heat production by eddy currents the armature is taken in a laminated form.

The manner in which lamination reduces the strength of the eddy current and thus reduce the loss of power due to such current can be realised from the following considerations. Iron is electrically conducting and the rotation of a solid core in a magnetic field induces eddy current which flows at right angles to the field through the core having electric paths of low resistance and there is consequent production of considerable current. If the core is subdivided into n thin sheets insulated from one another, it has the effect of reduction of cross-section of electrical paths increasing the resistance by n-times of each eddy current path and at the same time reducing the *emf* round an eddy current path to $\frac{1}{n}$th of its previous value. This reduces the eddy current loss by $1/n^2$ fraction. Solid and laminated cores are shown in Fig. 8·4-b.

Field Coil excitation : Field magnets in small machines are sometimes permanent magnets but generally these are self-excited electromagnets. The current produced in the machine is used back for exciting the field coils. At start the armature coil finds itself in a magnetic field, though weak, arising out of residual magnetism and a small *emf* is developed when the armature rotates. The current thus generated circulates the field coils making the field stronger and the machine very soon develops the requisite field and the maximum *emf*. There are three types of winding the field coil in relation to the armature coil.

(a) SERIES-WOUND MACHINE : In this mode of

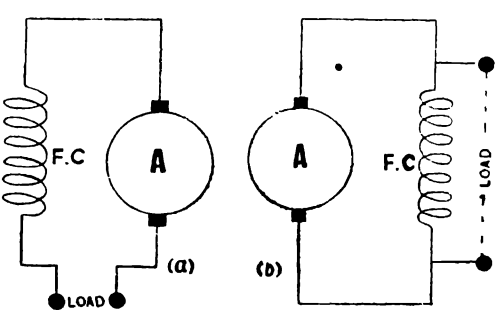


Fig. 8·5

winding (Fig. 8·5-a), the field coil consists of a few turns of

thick wire. The armature conductors, the field coil and the external circuit are all in series. If the external circuit, *i.e.* the load draws more current, the field coil current also increases and thus causes a higher voltage to generate.

(b) SHUNT-WOUND MACHINE : The field coil consists of large number of turns of wire joined in parallel with the armature coil and the external circuit (Fig 8·5b). As the load draws more current, the field coil current drops to a lower value. But if the terminal potentials be kept constant, the field coil current becomes independent of the load current.

(c) COMPOUND-WOUND MACHINE : It is a shunt-wound machine having in addition few turns in series. If

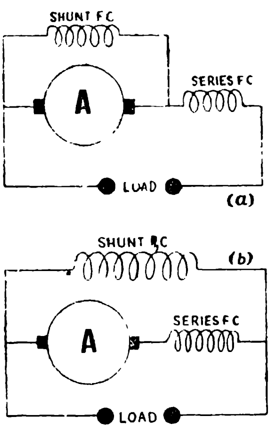


Fig. 8·6

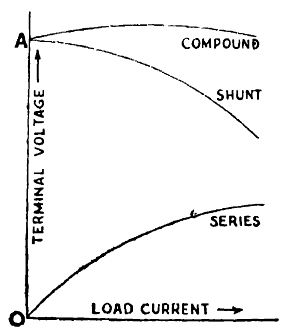


Fig. 8·7

the series coil be in series with the field-coil-armature parallel combination it is called a *short shunt* (Fig. 8·6-a) and if it is in series with the armature it is called a *long shunt* (Fig. 8.6-b). With this contrivance the terminal voltage may be kept practically constant irrespective of the current in the external circuit. The characteristics of three types of D.C. dynamo having different modes of field coil winding are shown graphically (Fig. 8·7).

Armature E.M.F. and Commutator action : There are two types of armature, Gramme-Ring armature and Drum arma-

ture. Of this the first is not much in use, but its action gives a clear conception regarding the cumulative effect of *emfs* in different coils or condutors.

GRAMME-RING ARMATURE : The armature core is an iron ring (R in Fig. 8·8) made up of sheet iron stampings and is wound uniformly with insulated copper coils. The coils though several in numbers yet are connected to one another in such a way as to form an endless coil. The ends of each coil are connected with two adjacent bars, a number of which are fitted insulated from one another around a slip-ring (C) called the commutator. The ring is fitted on the main shaft so that it rotates along with the armature core. Two metal brushes (B_1, B_2) are so fixed that they always graze the surfaces of the two copper bars in opposite sectors. As the commutator ring rotates each of the brushes comes in contact with one bar after another. Each brush is so placed that each of them comes in contact with the end of the coil when it is in a vertical position, *i.e.* when there is no *emf* in it. This is meant for avoid sparking that may arise due to short-circuiting of a coil through the brush when two consecutive copper bars come under the brush at the same instant (Fig. 8·9).

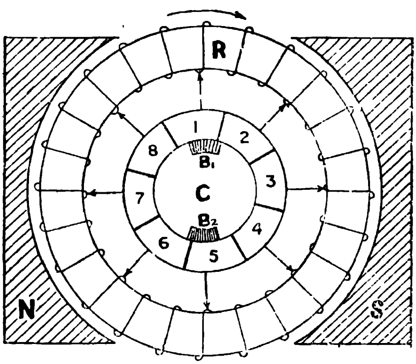


Fig. 8·8

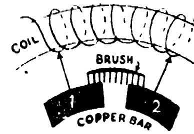


Fig. 8·9

When two brushes are at the same time in contact with two remote copper bars, they form the terminals of two parallel paths made by the coils on the respective halves of the armature. As the armature rotates the induced *emf* in the coils on each half lying under the same pole are in the same direction. But those on the two

halves are in opposition and they become cumulative when directing current in an external circuit through the brushes. The *emf* in each coil is sinusoidal but the *emfs* in different coils differ in phase due to difference in position with respect to the field. If e_1, e_2, e_3, e_4 be taken to be the *emf* in coils in different positions, the total *emf* at any instant in each half of the coils is obtained as

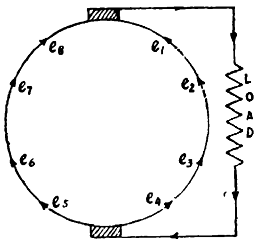


Fig. 8·10

$$e = e_1 + e_2 + e_3 + e_4 + \cdots\cdots\cdots$$

$$\text{and} \quad e = e_5 + e_6 + e_7 + e_8 + \cdots\cdots\cdots$$

Although the *emf* in each coil is changing during rotation, the sum of the *emfs* of the different coils remains constant, since position of the coils as a whole with respect to the field remains unaltered. So the machine develops almost a steady *emf* between the brushes and it sends an undirectional steady current through the external circuit.

Disadvantages of Ring armature : It may be observed that there is no field in the space encircled by the ring. As such one-half of the wire-turns does not embrace any flux even when the ring rotates. These are unproductive so far as the *emf* is concerned and act merely as return of the circuit. Further, there is wastage of electrical energy due to flow of current in these. Moreover winding is inconvenient and commutation difficult. These disadvantages have been removed in another type of armature.

DRUM ARMATURE : A drum-shaped core is made up of soft iron sheets insulated from one another. There are parallel slots cut lengthwise along the surface of the cylinder shaped armature. Conductors are housed inside the slots. Each conductor is connected directly to a second conductor on the backside and on the front side conductors are connected one after another through commutator bars so as to form a

complete circuit when joined with external circuit through brushes pressing on commutator bars.

The diagram (Fig. 8·11) shows in cross-section a four-pole dynamo. The conductors are indicated as black circles. The *emf* induced in each of the conductors under *N*-pole is in the same direction and that in the conductors under *S*-pole is in opposite direction relative to the other. These are indicated as '+' or '−' in the diagram. As the armature rotates a particular conductor is shifted from one pole to another and the *emf* continually changes in magnitude and alternately in direction. The distribution of *emf* in space, *i.e.* in conductors under a particular pole remains unchanged which is as considered outward under *N*-pole and inward under *S*-pole in the arrangement shown in the diagram drawn.

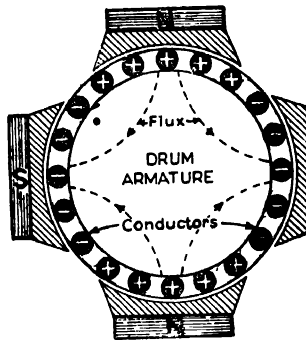


Fig. 8·11

There are two types of winding known as lap (or parallel) winding and wave (or series) winding. The difference in two types of winding lies in the arrangement of connections of one conductor with another ahead of it.

If the periphery of the armature as well as that of the yoke be opened we get the conventional 'developed' diagram in which the poles and the conductors are shown in one plane (Fig. 8·12).

If there are four poles and Z conductors in all, $Z/4$ of them lie under each pole. The *emf* in conductors under *N*-pole is opposite to those in the next group under *S*-pole.

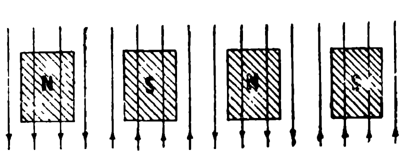


Fig. 8·12

The *pole-pitch* is the distance from one pole-centre to the next reckoned by the number of conductors intervening. In the case of a *p*-pole

generator having Z-conductors the pole-pitch is Z/p. The separation of the consecutive end-connections at the back and the front reckoned in term of number of conductors are respectively known as *back-pitch* and *front-pitch.*

Commutator bars insulated from one another are arranged round a ring fitted with the main shaft along with the drum. These bars are used for making forward corrections. There are half as many bars as the number of conductors. There are brushes pressing on the bars spaced in such a way as to form contacts at the extreme ends of the group of conductors in series forming a coil and generating cumulative *emf.*

Lap and Wave windings : A conductor, say no. *a* (Fig. 8·13) under a particular pole is to be connected at the back to a conductor, say no. *b*, under the next pole so that the *emfs* in them become cumulative. The conductor is now to be connected at the front with any other conductor. Such front connections can be made in two ways, backward or forward, *i.e.* with a conductor under a pole behind or ahead of it. The two methods involving backward and forward connections are classified as *lap and wave windings*. Whatever may be the principle adopted the connections should be such that the conductors one after another form a continuous path, the *emfs* being cumulative.

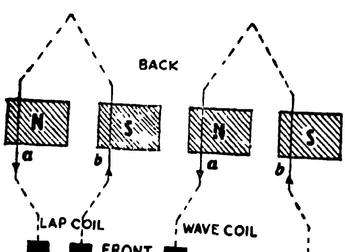


Fig. 8·13

The difference in connections between the two types of winding may be realised by remembering that in lap winding the front and back connections in a particular conductor are similar both either forward or backward, while in a wave winding the front and back connections of a particular conductor are dissimilar, one forward and the other backward. In alternate conductors in both the type of winding the connections are reversed. The number of conductors are so chosen as to form two or more parallel circuits.

LAP WINDING consists of a system in which the front and back pitches are unequal. The scheme of connections simplified with only 16 conductors under 4-poles in shown in diagram (Fig. 8·14).

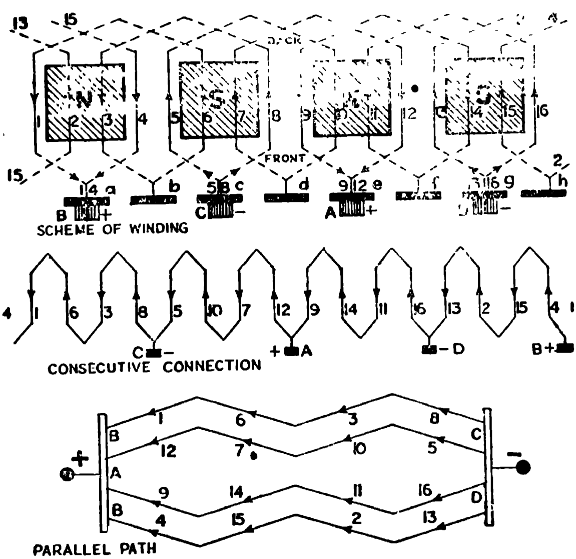


Fig. 8·14

Brushes for making connections of the conductors with the external circuit are placed at proper places. It may be observed from the diagrams that a brush is placed at each meeting or separating point of opposing *emfs*. The meeting points form the positive terminals and the separating points the negative terminals. According to this• the brushes *A* and *B* are positive and those at *C* and *D* are negative. There are four parallel paths (equal to the number of poles) as shown in the diagram.

WAVE WINDING is a system in which the front and back pitches are often equal. The scheme of connections simplified with 18 conductors are shown in the diagram (Fig. 8·15). The winding progresses in a series of waves and hence the name wave winding. There are only two parallel paths connecting all the conductors and two brushes are necessary.

But for equal spacing two more brushes are placed in positions shown in dotted lines. Thus there is a brush opposite to a pole.

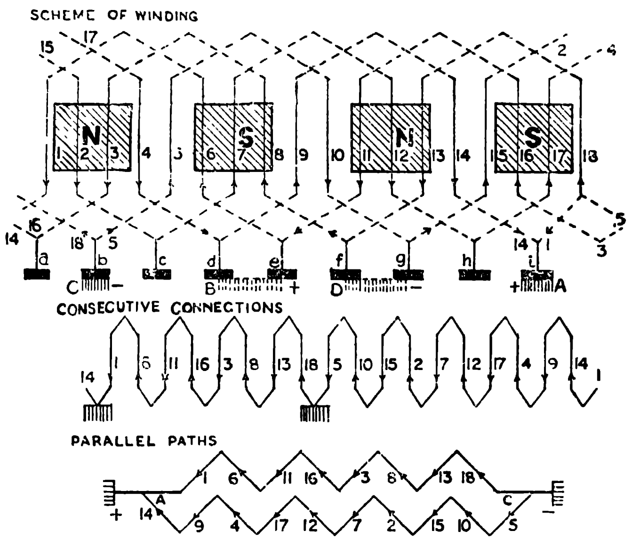


Fig. 8·15

Difference between Lap and Wave windings: The main points of dissimilarities between the two types of winding may be summarised as follows :

(*i*) In a wave winding considering the continuity of wiring it is found to be advancing both in the front and at the back. In lap winding at the back the wiring advances in the forward direction but in the front it is retrograde. This is sometimes expressed by saying that in the wave winding the front and the back pitches are in the same direction, in lap winding these are in opposite directions.

(*ii*) In a (single) wave winding there are two parallel paths through the armature coil irrespective of number of poles ; in a (single) lap winding there are as many paths in parallel in the armature as there are field poles.

(*iii*) In the wave winding two brush sets are only necessary and in lap winding the number of brush sets necessary is equal to the number of poles.

(iv) In a wave winding the front and back pitches are often equal but in lap winding they always differ by two.

(v) Wave winding is suitable for large voltage and lap winding for heavy current.

Practical form of windings : In a developed diagram the conductors are shown to be arranged uniformly round the periphery. To avoid practical difficulties of too many triangular connections at the ends of the conductors, these are taken in the form of coils. The coil sides are placed in slots in two layers, one upper and the other lower. Each coil lying with one side in the upper half of one slot has the other side in lower half of a slot one pole pitch apart.

To limit the number of slots, more than one conductor is placed in each layer. In large machines six or eight conductors is placed in one slot. The individual coil formed by a pair of conductors under consecutive poles form a single turn coil.

The number of segments or bars in the commutator ring should be half as many as the number of conductors. So an armature with small diameter should have very thin segments. To avoid this difficulty in small machines all the conductors in the upper layer in a particular slot together with those in the lower layer of the slot one pole-pitch away form a multiturn coil having two open ends only.

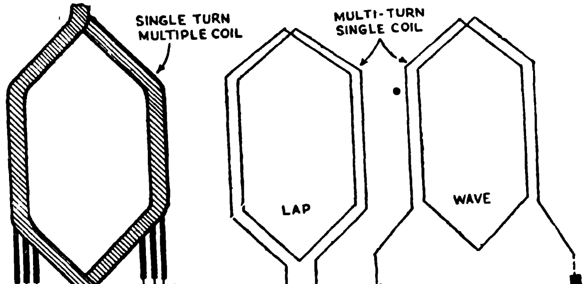


Fig. 8·16

The practical winding has the following differences with the schematic windings shown in the developed diagram :

(i) It is a two layer winding and as such has even number of coil sides per slot.

(ii) In it each individual conductor may be the coil side of a single turn or a multiturn coil.

An important requirement in a dynamo, that all the coils shall be identical can thus be obtained by using pre-formed coils placed in normal positions in the slots. What appears to be a single coil in the armature may be actually several independent single turn coils or a multiturn individual coil (Fig. 8·16).

ARMATURE REACTION : Armature has an iron core and the conductors round it carry current. The distribution of armature current is equivalent to a solenoid carrying current and the armature sets up a field at right angles to the main field. If OC (Fig. 8·17) is the main field and OD due to armature reaction the resultant field is along OF. This causes a twist of the field in a direction in which the armature rotates.

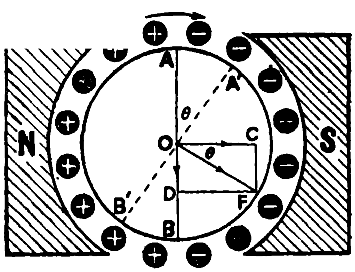


Fig. 8·17

Again, the brushes are to be placed at right angles to field to avoid sparking which happens due to short-circuiting of a particular coil. So due to this distortion of the field the brushes are to be shifted from geometrical neutral axis (AB) to the magnetic neutral axis ($A'B'$). The angle through which the brushes are shifted is called the *angle of lead* (AOA' in the diagram). If the brushes are given a lead the effect of cross-magnetisation is remedied but demagnetising effect which weakens the field still exists. These two effects of cross-magnetisation and demagnetisation are called ***armature reaction.***

Instead of shifting the brush position the armature reaction is annulled by small extra poles (called inter poles) excited by the armature current in series placed midway between two main poles and having the same polarity as the next

main pole in the direction of rotation. Alternativaly neutralising winding in series with the armature may be placed inside the material of the pole pieces.

Expression for E.M.F. : Let us suppose that there are several conductors under each pole and *emf* in them at any instant are respectively e_1, e_2, e_3....... The *emfs* in the conductors are additive in each of the parallel paths. Hence the total *emf* generated in all the conductors under the same pole is

$$e = e_1 + e_2 + e_3 + \cdot$$

e_1, e_2, e_3 . are all different because of their positions in the field.

Let N be the flux per pole. Divide it as N_1, N_2, $N_3 \cdots$ in as many parts as there conductors in it. Let x be the spacing between consecutive conductors and t the time required by a conductor to move through this distance. Induced *emf* in the conductor embracing this flux N_1 is N_1/t and considering similar *emfs* in other conductors, we have

$$e = \frac{N_1}{t} + \frac{N_2}{t} + \frac{N_3}{t} + \cdots\cdots$$

If n be the revolutions per second, angular velocity $\omega = 2\pi n$. If in all there are Z conductors round the armature, x subtends an angle $2\pi/Z$ at the axis of revolution and t is obtained as

$$t = \frac{2\pi}{Z} \div 2\pi n = \frac{1}{nZ}$$

Therefore *emf* induced in all the conductors under one pole is

$$e = \Sigma \frac{N_1}{t} = ZnN$$

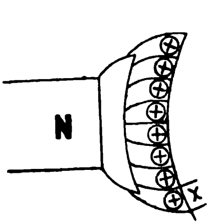


Fig. 8·18

If there are P poles and if all the conductors are in series the total *emf* $E = PNZn$. But if there are p parallel paths through the P poles,

$$\text{Total } E.M.F. \quad E=\frac{NZnP}{p}\times 10^{-8} \text{ volts.}$$

$p=P$ for lap winding, and $p=2$ for wave winding.

ILLUSTRATIVE EXAMPLE

The armature of a 4-pole D.C. generator has wave winding consisting of 774 conductors. Calculate the emf generated when the flux per pole is 4×10^{-6} maxwells and the speed of revolution is 250 r.p.m.

Solution : $E=\frac{NZnP}{p}\times 10^{-8}$ volts

$N=4\times 10^{-6}$, $Z=774$, $n=250/60$, $P=4$, $p=2$

Hence $E=\frac{4\times 10^{-6}\times 774\times 4\times 250\times 10^{-8}}{2\times 60}$

or $E=258$ volts.

VOLTAGE REGULATION : The voltage in a generator is regulated by controlling the field current by means of a rheostat placed in series with a shunt-wound generator and used as a shunt in a series wound machine.

In a shunt-wound generator (Fig. 8·19a) the more the resistance (R) placed in series with the field coil the less is the exciting current in the field magnet. This causes a lowering of the voltage generated. Maximum voltage is obtained when $R=0$.

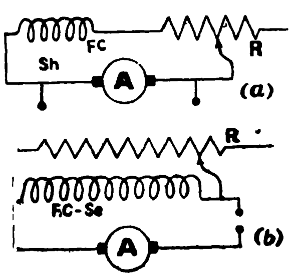


Fig. 8·19

In a series-wound machine a resistance (R) is put in parallel with the series field coil. If R is diminished, greater current is diverted through R, diminishing the field coil current. This causes a fall of voltage. The voltage is maximum when R is infinity *i.e.* when R is disconnected from the circuit.

Efficiency of the Generator : The efficiency is considered in three different ways for different purposes.

$$\text{Commercial Efficiency} = \frac{\textit{Power delivered to the load}}{\textit{Power supplied to the machine}}$$

$$\text{Electrical Efficiency} = \frac{\textit{Power in the external circuit}}{\textit{Total power generated}}$$

$$\text{Mechanical Efficiency} = \frac{\textit{Total power generated}}{\textit{Power supplied to the machine}}$$

Input in a machine is the power supplied to it. Iron and frictional losses cause a lowering of this.

Power developed—copper losses = Power delivered (output)
Input—Iron and mechanical losses = Power developed
= Output+copper losses

Losses in a D.C. generator may be summarised as shown below.

Total loss = Copper loss+Iron loss+Mechanical loss
Copper loss = Loss in armature core+loss in shunt coil +loss in series coil
Iron loss = Hysteresis loss+Eddy current loss.
Mechanical loss = Frictional loss+air loss in rotation.

VIII-3. A.C. GENERATORS

Essential parts : In a modern A.C. dynamo the armature remains stationary and the field is made to rotate in the space inside the armature. A stationary armature simplifies the construction and the problems regarding insulation. This form of armature eliminates the necessity of commutators and an alternator constructed as such may be made to supply large output at high voltage. The field is excited by supplying direct current to the windings from a small D.C. generator. A pair of slip rings brings this current to the rotating poles.

The stator core is laminated. It is not necessary to laminate the rotor core, since the current in the windings round the rotor core is continuous. Sometimes pole-tips are taken in laminated form. The general construction of an alternator is described below and is shown in the diagram (Fig. 8·20).

(i) The stationary armature or stator as it is called, is built of steel laminations having slots for conductors. It is fitted inside a cast iron frame.

(ii) The rotor poles are fixed to the rim of a steel magnet wheel which can be rotated. The exciting coils round the poles are connected in series and the two ends are connected to the D.C. source through brushes.

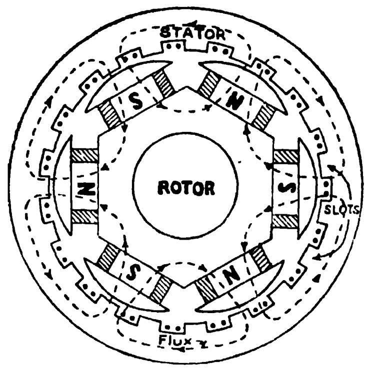


Fig. 8·20

(iii) The armature coils are fitted inside the slots their both ends being brought out. For a maximum voltage the span of each coil should be a pole-pitch but for improving the wave shape of the voltage generated the span of the coil is reduced.

Single-phase winding : There is only one coil group per pair of poles. In *concentrated winding* there is one slot per pole. This type of winding gives the maximum voltage for a given number of conductors but the wave form is not sinusoidal. Better wave form is obtained by distributing the windings in several slots per pole.

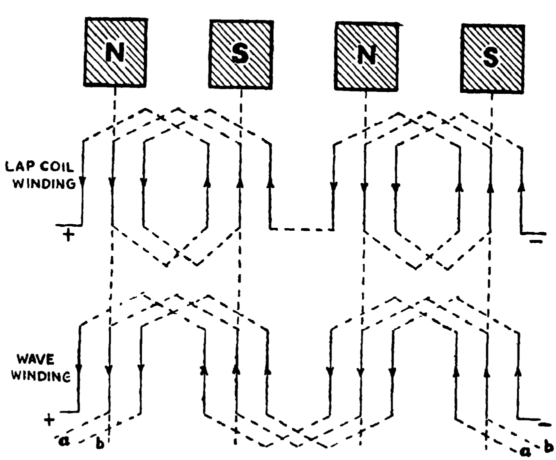


Fig. 8·21

The windings may be of lap or wave pattern. All the conductors or sets of coils are connected in series and the two ends

are joined to the terminals of the machine. The diagram (Fig. 8·21) shows mode of *distributed* lap or wave windings.

The windings shown are single layer windings. Two-layer windings arranged with two coil sides per slot, exactly like D.C. armature windings, are frequently used.

Poly-phase winding : The windings in an A.C. generator may be so arranged that instead of a single circuit from the machine, there are two or three separate circuits supplying *emf* of the same frequency and voltage but differing in phase by 90° or 120°. These are respectively called two-phase and three-phase machines.

In a **two-phase alternator** two separate single phase winding are housed in slots in different positions relative to the poles in such a way that *emfs* in the two circuits differ in phase by 90°. The conductors in one set are placed half-way between the conductors of the other set so that they are 90° electrical degrees apart. This means that when the *emf* in one set is maximum that in the other is zero. There are two sets of terminals of the machine. A skeleton two-phase winding is shown in the diagram (Fig. 8·22 upper portion).

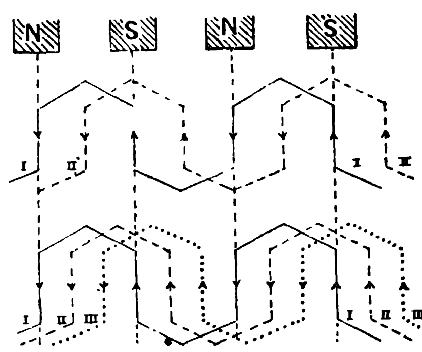


Fig. 8·22

In a three-phase alternator three sets of conductors are distributed in slots, each set occupying one-third of each pole-pitch. By this arrangement each set of conductor is placed 60° apart in phase from the other. The actual phase difference between the *emfs* generated in the first and second set is 60°, between second and third set is 60° and that between first and third set is 120°. If the middle set of conductors is reversed in connection with the terminals the actual phase difference between any two of the sets is 120°. A skeleton winding is shown in Fig. 8·22 (lower portion).

Frequency Equation : The *emf* in each conductor goes through a complete cycle when it passes through an angular distance equal to twice the pole pitch, *i.e.* it moves past one pair of poles. If there be n revolutions per second and $P/2$ is the number of pairs of poles, the number of complete cycles in one second, *i.e.* the frequency of *emf* is obtained as

$$f=n.\frac{P}{2}$$

If again R is the revolution per minute $n=R/60$, hence

$$f=\frac{R}{60}\cdot\frac{P}{2}=\frac{R.P}{120}$$

Thus if speed and frequency are fixed, the number of poles required becomes fixed. For example if $f=50$ and $R=1500$ *r.p.m*, the number of poles will be

$$P=\frac{120f}{R}=\frac{120\times 50}{1500}=4.$$

E.M.F. equation : Let N be the flux per pole in maxwells, P is the number of poles, n is the speed in revolution per second and f the frequency. Let us consider the case in which there are Z conductors in series per phase.

Time taken by the field for one revolution $=\frac{1}{n}$ sec

Time taken by the field to move through one pole pitch is

$$t=\frac{1}{Pn}$$

During this interval each conductor is cut by the entire flux of one pole and sc the average *emf* in each conductor is

$$e=\frac{N}{t}=PnN\times 10^{-8} \text{ volts.}$$

R.M.S. value=average value $\times$ form factor and for a sinusoidal *emf* form factor is 1·11. Hence for an alternator generating such an *emf*

R.M.S. emf per conductor $=1{\cdot}11\times PNn\times 10^{-8}$ volts. Hence considering all the Z-conductors, the resultant (*R.M.S.*) *emf*

$$E=1{\cdot}11\times PNn\times Z\times 10^{-x} \text{ volts.}$$

If the wave form is not sinusoidal the form factor differs from 1·11. Indicating this by K_1 the equation for the *emf* should be expressed as

$$E = K_1 PNnZ \times 10^{-8} \text{ volts}$$

If the windings are distributed then the *emfs* in the various parts of the coil are not in phase. Suppose that the total *emf* in the coil for concentrated winding is E and for distribution it is reduced to K_2E, then K_2 is called the *'breadth factor'*. Hence the resultant *emf* in all the conductors is expressed as

$$E = K_1 . K_2 PNnZ \times 10^{-8} \text{ volts}$$

Since frequency $f = \frac{nP}{2}$, $E = 2K_1 K_2 fNZ \times 10^{-8}$ volts

Terminals in a three-phase winding : Since a three-phase alternator has three separate circuits supplying current in three different phases there should be six terminals of the machine. But in practice there are two general methods in which the number of conductors necessary reduces to three only. These are *star* and *delta* (*or mesh*) connections.

STAR CONNECTION : The three phases each have one end brought to a common point called the *neutral* or *star point*, while the other ends are taken to three terminals of the machine. The *emf* between any two phase lines is called *line voltage* (E_L). It is equal to the vector difference of the two phase voltages (E_p), hence

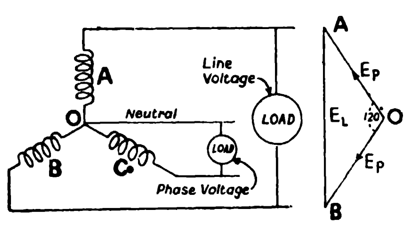


Fig. 8·23

$$E_L = \sqrt{E_p^2 + E_p^2 - 2E_p . E_p . \cos 120^\circ}$$

$$\text{or} \quad E_L = E_p \sqrt{2 + 2 \cos \frac{\pi}{3}} = \sqrt{3} E_p$$

So Line voltage $= \sqrt{3}$. (phase voltage). There is a phase difference of 30° between the two. In a practical system of

supply the phase voltage is made 230 volts and in such a case the line voltage becomes 400 volts.

In practice, the phase voltage is supplied for domestic consumptions and for this purpose a neutral line (earthed) from the star point runs along with the phase lines, and the phase voltage is obtained by connecting the house supply mains with a phase line and a neutral line. The neutral line is not necessary in transmission lines where all the three phases are utilised. Since each line is connected to one phase only in star-wound machine the line current is same as the phase current.

DELTA CONNECTION : In this system the three phases are joined to form a circuit which is closed. Since the sum of the instantaneous values of three *emfs* is always zero, there is no current circulating round the mesh. Between each pair of lines, there is only one phase *emf* and so phase *emf* is same as the line *emf*. The line current is the vector difference of two phase currents fed into a line.

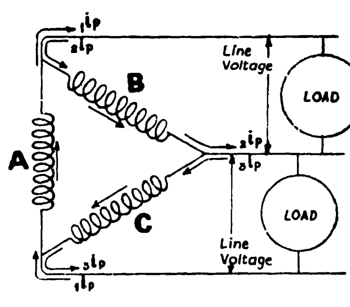


Fig. 8·24

Hence the line current is $\sqrt{3}$. (phase current), *i.e.* $i_L=\sqrt{3}i_p$.

Power in three-phase system : Since a balanced three-phase system consists of three similar circuits,

$$\text{Total power}=3\times(\text{power in each phase})$$

$$=3\times E_p i_p \cos\phi$$

ϕ is the angle between the phase *emf* and phase current. Since in star connection $E_p=E_L/\sqrt{3}$ and in delta connection $i_p=i_L/\sqrt{3}$, so in both systems

$$\text{Total power}=3\times\frac{1}{\sqrt{3}}E_L i_L \cos\phi=\sqrt{3}E_L i_L \cos\phi$$

It should be noted that ϕ is the ***phase difference between phase current and phase voltage.***

ILLUSTRATIVE EXAMPLES

1. A star-wound alternator develops a line emf of 6600 volts. Calculate the phase voltage and the total power output when the current in each phase is 300 amperes and the power factor is 0·8.

Solution : Phase voltage $E_p = \dfrac{6000}{\sqrt{3}} = 3810$ volts.

Power output $= 3 \times E_p i_p \cos\phi = 3 \times 3810 \times 300 \times 0{\cdot}8$

$= 2740 \times 10^3$ watts.

2. A 100 H.P. three-phase delta-connected motor works on a line voltage of 3000 volts. If the efficiency is 84 percent and power factor 80 percent, calculate the line and the phase currents.

Solution : Input $=$ output $\times \dfrac{100}{84}$

$$= \frac{100 \times 746 \times 100}{84} = 88 \times 10^3 \text{ watts}$$

Power $P = \sqrt{3} E_L i_L \cos\phi$

$$\text{or} \quad i_L = \frac{P}{\sqrt{3} E_L \cos\phi} = \frac{88000}{\sqrt{3} \times 3000 \times 0{\cdot}8} = 21{\cdot}17 \text{ amp.}$$

VIII-4. D.C. MOTORS

Principle : An electric motor is a contrivance to convert electrical energy into mechanical energy. If a conductor of length l carrying a current i (e.m.u.) is placed at right angles to a magnetic field (H) it is urged on by a force Hil dynes according to Fleming's left hand rule. Consider that the armature of a D.C. generator instead of being rotated by mechanical power is supplied with current through its conductors from an external source. Force acts on each of the conductors and the force on the conductors in one-half of the armature facing a particular pole is equal and opposite to that on the other half facing the opposite pole. This produces a couple about the axis of the armature which causes the armature to rotate. The machine now acts as a motor. The components of such

a motor is similar to that of a generator (§ VIII-2), comprising the field magnet, the armature and the commutator. A steady current is supplied both to the armature and the field coils. The field coil current may be, as in a generator, in series, in parallel or in both forms in relation to armature coils.

Torque equation : Let N_o be the flux distributed around each conductors of length l and let the space occupied along the periphery by each of the conductor be x in length. Thus the field H (measured as flux per unit area) in which a conductor is placed is given by

$$H=\frac{N_o}{Lx}.$$

Let r be the radius of the armature and Z be the total number of conductors then $x=2\pi r/Z$.

Suppose i_a is the total armature current in p parallel paths then the current in each conductor is i_a/p. So the force on each conductor obtained as Hil is given by

$$\frac{N_o}{Lx}\cdot\frac{i_a}{p}.\, l=\frac{N_o.Z}{2\pi r}\cdot\frac{i_a}{P}.$$

If N is the flux per pole, the total force on all the conductors under each pole is

$$F=\Sigma\frac{N_oZ}{2\pi r}.\frac{i_a}{p}=\frac{NZ}{2\pi rp}.i_a$$

Torque acting on the armature due to this if r be the radius of the armature, is given by

$$F.r=\frac{NZi_a}{2\pi rp}.r=\frac{NZi_a}{2\pi p}.$$

If there are P poles, the total torque on the armature is

$$T=\frac{NZi_a.P}{2\pi p}$$

$$\text{or}\quad T\propto Ni_a$$

Back E M.F. : When the armature begins to rotate there is induced *emf* in the coils and the motor functions as a

generator. The induced *emf* in the coils is in a direction opposite to that supplied to them. This is called ***back emf in a motor.*** The effective current through the armature is less than that supplied as it is diminished by the back *emf.*

Resultant *emf* in armature = Supply *emf* — Back *emf*

If R_a is the armature resistance and i_a the current in it, V the *emf* supplied and E is the back *emf*, then

$$R_a i_a = V - E$$

Speed equation : Let f be the frequency of rotation of the armature. The magnitude of the back *emf* is controlled by the factors determining the *emf* on a generator and as such $E \propto Nf$ (see § VII-2). So

$$\text{Frequency} \quad f \propto \frac{E}{N}.$$

$$\text{But} \quad E = V - R_a i_a \text{, so } f = \frac{V - R_a i_a}{N}$$

Speed of a motor, as shown, varies directly as the back *emf* and inversely as the flux. So the speed can be increased by weakening the field. It is thus seen that the condition for a large torque ($T \propto Ni_a$) is different from the condition of high speed ($f \propto E/N$). Speed may be varied by either varying the applied voltage (V) or the field flux (N).

EFFECT OF LOAD : A motor is said to be loaded when it overcomes any external torque opposing its motion. The appliance which is to be driven when coupled with motor creates such a counter torque. When a motor is set in motion without having any load, the torque in motor is required only to overcome friction. In such a case back *emf* rises to a value very nearly to the supply voltage. When a load is applied, the motor tends to fall in speed, the back *emf* decreases causing the armature current to increase to such a value as may produce the torque necessary to overcome the counter torque produced by the load. If the load is decreased, the existing torque in the motor being in excess accelerates the armature, which again increases the back *emf* till the

current falls to new value necessary to overcome the diminished torque created by the new load. The back *emf* thus makes the motor a self-adjusting machine, regulating the current to such a value as is necessary for the applied load.

Field excitation : There are three ways of exciting the field of a D.C. motor. Excitation involves the supply of current to the field and the armature coils, which is done by either series, shunt or compound form of windings.

SERIES MOTOR : The field coil is in series with armature coil, and the same current flows through the both. The flux increases with the current and speed falls with rise of current, since speed is inversely proportional to flux.

At light loads the back *emf* causes the current to be small which makes the field weak. So the speed which is inversely proportional to flux is high. The speed in such a case may

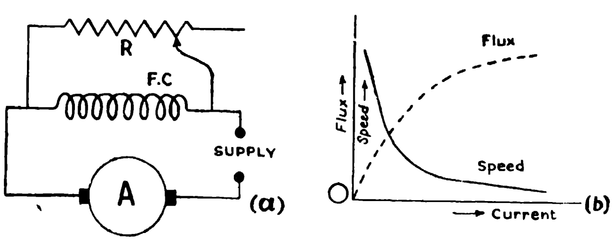


Fig. 8·25

be dangerously high. If the load increases causing rise of field strength, the speed falls. The speed at the start is inversely proportional to the current and the speed-armature current curve is a rectangular hyperbola (Fig. 8·25b). Since the back *emf* is not set up until some speed is attained, the starting torque is powerful. So series motors are suitable for work requiring a big starting torque as in setting tram cars or electric trains in motion from rest.

Because of high speed at light loads series motors are not run without some mechanical load and is coupled to the load through gears and not by belts, which may be torn to pieces when the speed is high.

SHUNT MOTOR : In this type of motor the field coil is in parallel with armature coil so the current in the field coil is constant and flux is independent of the load. The motor runs at a constant speed. Shunt motor are used for any drive which requires a constant speed as in driving of lathes. The speed-current curve is shown in Fig. 8·26a.

COMPOUND MOTOR : This kind of motor (Fig. 8·26b) has both shunt and the series coils between the field and the armature windings. It exerts a large starting torque but does not acquire a high speed immediately with the removal of the load. The motor is therefore used in machineries where a

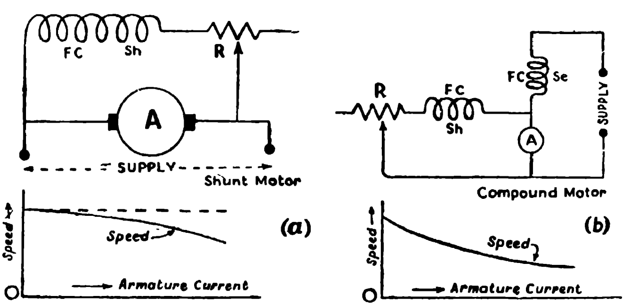


Fig. 8·26

large starting torque is necessary and the load may be thrown off suddenly. By adjusting the relative strengths of shunt and series coils, the compound motor may be given any characteristic between those of shunt and series motors. The shunt and series coils may be arranged to help one another (cumulative method) or they may be in direct opposition (differential method).

Speed Regulation : The speed of a motor is obtained from the equation $f=(V-i_a R_a)/N$. So the speed may be regulated by controlling either V or N. The flux per pole depends on the strength of exciting current and thus if the current is varied the speed will vary. Since speed increases for decrease of flux increased speed is obtained by decreasing the exciting current or by decreasing the ampere-turns.

In a *series motor*, the field current is diminished by a

diverter in the form of a resistor in parallel with the field coil (R in Fig. 8·25a).

In a *shunt motor* to regulate the speed a rheostat is placed in series with the field coil that is with the shunt-winding (R in Fig 8·26-a).

In a *compound motor* a regulating resistance is placed in series with the shunt coil (R in Fig. 8·26b).

Speed may also be controlled by varying the supply voltage. A variable resistance in series with the armature serves this purpose.

Motor Starters : When a voltage is applied to the motor the armature coil acts like a low resistance circuit and the current is generally high. When the motor acquires speed the back *emf* is developed and the current falls to a lower value Thus the current at the start may be so high as to cause a damage to the machine. To prevent this some arrangement is made which comprises what is called a *motor-starter*.

In series motor an adjustable resistance (R) is connected in series with the field coil. There is a starter switch whose arm (H) moves over the adjustable resistance R. The arm of the switch is so placed that at the start there is some additional resistance in the circuit to bring down the current in the armature and the field coil to low value so as to prevent damage. The starter resistance is gradually decreased till at last when the motion gains speed the entire resistance is cut off and the arm reaches the point of zero-resistance position. It is held there by an electromagnet excited by current through the motor. If the supply from mains fails at any instant, the electro-magnet ceases to remain operative and the starter arm is released and thrown back to the off-position by a spring attached to it.

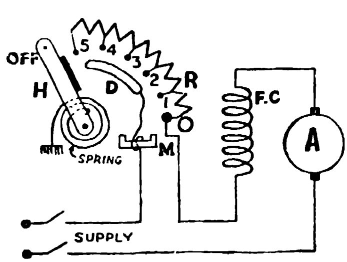


Fig. 8·27

In a *compound* or *shunt motor* an adjustable resistance is

put in series with the field coil. A movable arm (H) while moving over the adjustable resistance (R) causes gradually increasing current to flow through the motor, till at last in the zero-position of the arm the entire resistance is cut off. The arm is held there by an electro-magnet (M_1) excited by the field coil current. If the main supply stops the electromagnet becomes inoperative and the arm is drawn back by a spring to the off-position. Over and above the no-volt release there is an additional arrangement for over-load release. The electro-magnet (M_2) is energised by the full current supplied to the motor. If the current exceeds its safe value the electromagnet draws up a pivoted arm P which on being attracted makes contact with two studs short-circuiting the electro-magnet (M_1) coil. This makes M_1 inoperative and the arm being released makes the motor disconnected from supply mains.

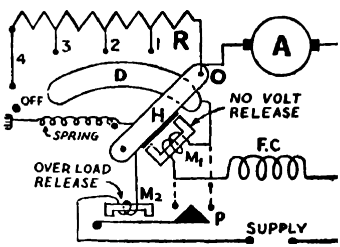


Fig. 8·28

VIII-5. ROTATING MAGNETIC FIELD

Mutually perpendicular sinusoidal fields : Let two coils C_1, C_2 carrying alternating currents of same amplitude and frequency (f) but differing in phase by $\pi/2$ be placed in mutually perpendicular planes. The two fields produced in the space between the coils represented as $H_0 cospt$ and $H_0 sinpt$ are at right angles to each other. The resultant field is given in magnitude by

$$H = \sqrt{H_0{}^2\, sin^2pt + H_0{}^2 cos^2pt} = H_0$$

and the phase angle ϕ is obtained as

$$tan\ \phi = \frac{H_0 sin\ pt}{H_0 cos\ pt} = tan\ pt$$

So $\quad \phi = pt = 2\pi ft.$

This change in phase indicates the rotation of the field with time. So it is obtained that the resultant field is of constant magnitude but its direction rotates with constant angular velocity $2\pi f$. As t increases ϕ also increases, so the rotation is anti-clockwise. It would be clockwise if the position of the coils are interchanged in position. If the fields have unequal amplitudes the resultant magnetic field varies in magnitude while rotating.

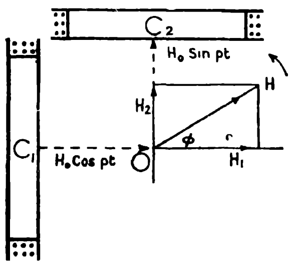


Fig. 8·29

Suoh an arrangement of producing a rotating magnetic field may be made by energising two identical coils placed in mutual perpendicular positions respectively by currents differing in phase by $\pi/2$ obtained from a two phase alternator.

Rotating field produced by three coils : Consider the resultant of three magnetic fields produced by three identical coils placed along the circumference of a circle at angular distances of 120° (which is the angle between the normals to the coils) from one another and each energised by currents differing in phase by $\frac{2}{3}\pi$ from the current in two others. Fields thus produced may be represented in magnitude as

$$H_0 sin\ pt,\ H_0 sin\ (pt-\tfrac{2}{3}\pi),\ H_c sin\ (pt-\tfrac{4}{3}\pi)$$

To obtain the resultant resolve the two other fields along and at right angle to $H_0 sinpt$. The components when

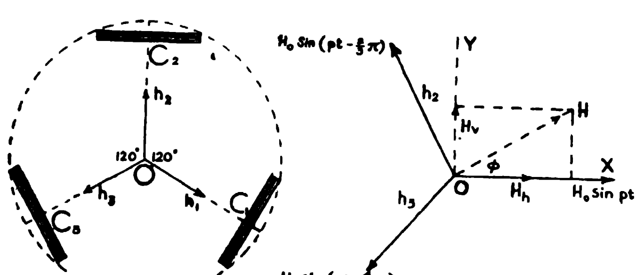


Fig. 8·30

added up represent the resultant fields as shown below.

Field along $H_0 \sin pt$, $H_h = H_0[\sin pt + \sin(pt - \frac{2}{3}\pi)(-\cos\frac{1}{3}\pi) + \sin(pt - \frac{4}{3}\pi)(-\cos\frac{1}{3}\pi)]$

or $H_h = \frac{3}{2} H_0 \sin pt$

Field at right angles to $H_0 \sin pt$,

$$H_v = H_0[0 + \sin(pt - \tfrac{2}{3}\pi)(-\sin\tfrac{1}{3}\pi) + \sin(pt - \tfrac{4}{3}\pi)(\sin\tfrac{1}{3}\pi)]$$

or $H_v = \frac{3}{2} H_0 \cos pt$

Resultant magnetic field is of magnitude given by

$$H = \sqrt{\tfrac{9}{4} H_0^2(\sin^2 pt + \cos^2 pt)} = \tfrac{3}{2} H_0$$

The field rotates with an angular velocity p obtained as change in phase ϕ given by

$$\tan\phi = \frac{\frac{3}{2} H_0 \sin pt}{\frac{3}{2} H_0 \cos pt} = \tan pt$$

or $\phi = pt = 2\pi ft$

f is the frequency of rotation of the field.

Couple on a coil in a rotating field : If a coil be placed in a rotating magnetic field, there will be induced *emf* in the coil which according to Lenz's law tends to oppose the relative motion of the coil and the field. As such the coil rotates due to couple acting on it in the direction of the field. The magnitude of the couple may be calculated for a plane coil as shown below.

Let the coil have an *effective area* A and let at any instant t the angular separation between the plane of the coil and the field (H) be ωt. ω is actually the difference of the angular velocity p of the field and p_0 that of the coil. The flux passing through the coil at any instant t is given by

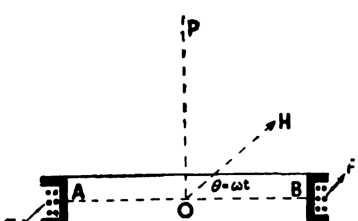


Fig. 8·31

$$N = AH \sin \omega t$$

The *emf* induced is $e = -\frac{dN}{dt} = -\frac{d}{dt}(AH\ sin\ \omega t)$

$$\text{or} \quad e = -AH\omega\ cos\ \omega t = AH\omega\ sin\left(\omega t - \frac{\pi}{2}\right)$$

Let L be the self-inductance and r the resistance of the coil. Its impedance $Z = \sqrt{L^2\omega^2 + r^2}$. The current in the coil is

$$i = \frac{AH\omega}{Z} sin\left(\omega t - \frac{\pi}{2} - \theta\right)$$

θ is the phase lag of the current behind the *emf* given by $\theta = tan^{-1}(L\omega/r)$.

The magnetic moment of the coil is Ai and it is directed along the normal OP of the coil. The couple tending to bring the normal in the direction of H is $AiH\ cos\ \omega t$. So the direction of motion of the coil is opposite to the direction of motion of the field. Regarded to be in the direction of rotation of the coil the magnitude of the couple (considered as negative) is obtained as

$$C = -AiH\ cos\ \omega t = = \frac{A^2H^2\omega}{Z} sin\left(\omega t - \frac{\pi}{2} + \theta\right) cos\ \omega t$$

$$= -\frac{A^2H^2\omega}{Z}[sin\ \omega t\ cos\ (\pi/2+\theta) - cos\ \omega t\ sin(\pi/2+\theta)]\ cos\ \omega t$$

$$= -\frac{A^2H^2\omega}{Z}[sin\ \omega t\ cos\ \omega t\ cos(\pi/2+\theta) - cos^2\omega t . sin(\pi/2+\theta)]$$

Taking the average over a complete cycle

$$C = +\frac{A^2H^2\omega}{2Z} sin\left(\frac{\pi}{2} + \theta\right) = \frac{A^2H^2\omega}{2Z} cos\theta$$

$$\text{Since} \quad tan\theta = \frac{L\omega}{r}, \quad cos\theta = \frac{r}{\sqrt{L^2\omega^2 + r^2}}$$

$$\text{Substituting for } Z, \quad C = \frac{A^2H^2\omega r}{2(L^2\omega^2 + r^2)}$$

The mean couple is in the direction of rotation of the field as indicated by the positive sign. Its magnitude depends upon ω, the relative angular velocity of the field and the coil.

Let us consider that the coil is mounted in such a way that it is free to rotate. When appropriate current passes through the field coils, the movable coil begins to rotate in the same direction as the rotating field. The angular velocity of the coil increases and value of ω decreases. But the instantaneous couple ($C=-AiH\cos\omega t$) still further increases. The angular velocity (p_o) of the coil goes on increasing until the rate at which the work is done by the rotating field is equal to the work done by the coil in rotating in opposition to frictional forces. The value of ω for which the average couple becomes maximum is obtained as shown below.

$$C = \frac{A^2H^2r\omega}{2(L^2\omega^2+r^2)}$$

$$\frac{dC}{d\omega} = A^2H^2r.\frac{L^2\omega^2+r^2-2L^2\omega^2}{2(L^2\omega^2+r^2)^2}$$

Putting $\frac{dC}{d\omega}=0$, we get $L^2\omega^2 = r^2$

or $\omega = \frac{r}{L}$

Again $\tan\theta = \frac{L\omega}{r}$, Hence the value of θ is $\pi/4$ for a maximum couple. The value of the couple under this condition is $\frac{A^2H^2}{4L}$.

The fact that a closed coil when placed in a rotating magnetic field experiences a couple is applied in practice in the construction of a type of motor known as *induction motor*.

VIII-6. A.C. MOTORS

Induction Motors : This type of A.C. motor works on the principle of a motion of a coil produced when it is placed in

a rotating magnetic field. These motors are constructed to work on two-phase, three-phase or single-phase current supply.

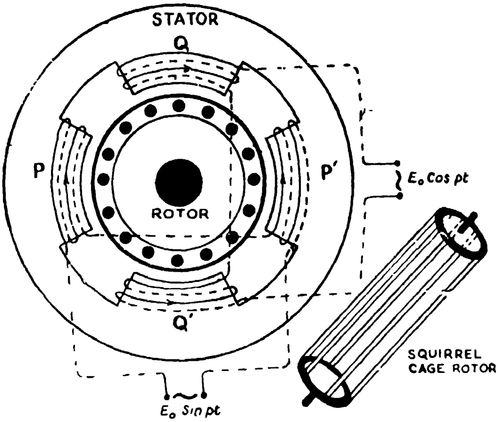


Fig. 8·32

TWO PHASE MOTOR : Two pairs of pole pieces *P-P'* and *Q-Q'* are placed inside and at the ends of mutually perpendicular diameters of a soft iron ring Individual phase currents differing in phase by $\pi/2$ from a two-phase A.C. supply are passed through the coils of each of the pairs of poles. The currents in each pair of coils are so directed that in between them there are two alternating magnetic fields. Due to action of mutually perpendicular sinusoidal fields a rotating magnetic field is produced inside the ring which is of constant magnitude but makes a complete revolution during a complete cycle of alternating current supplied.

If a conducting rectangular loop free to rotate is mounted inside the ring a current will be induced in it and the coil is acted upon by a torque which causes it to rotate in the same direction as the field. This is the motor action. It may be noted that the rotoı itself is not supplied with any current from external source.

Rotors : In a practical form of induction motor, the rotor consists of a core or drum of laminated soft iron discs of high permeability. There are slots in the drum in which copper bars are placed these being short-circuited at the ends by rings of copper. This type of rotors is called 'squirrel cage' rotor. In these the windings are permanently short-circuited and 'starter resistance' cannot be joined. This type of rotor is used only for small motors. There is another type known as

'phase wound' rotors. These are provided with distributed winding and for starting purpose they are connected with starting resistance.

SINGLE PHASE MOTOR : The field produced by a single-phase winding is alternating but it is not a rotating one. A single alternating field may be looked upon as the resultant of two equal fields rotating with equal angular velocity in mutually opposite directions.

Let two rotating fields of magnitude H_0 and angular velocity p coincide at any instant along OX (Fig. 8·33). After an interval t one field advances in the positive direction through an angle pt and the other in the negative direction from OX. The components along OY are respectively $H_0 cos(\pi/2-pt)$ and $H_0 cos(\pi/2+pt)$. These two mutually cancel. The components along OX are $H_0 cos\, pt$ and $H_0 cos(-pt)$, the latter being equal to $H_0 cospt$. Hence the resultant along OX is $2H_0 cos\, pt$. This proves that a single alternating field in any direction may be considered as the resultant of two equal fields rotating in mutually opposite directions. The component fields have the same frequency as the resultant but amplitude of each of them is half of the resulting field.

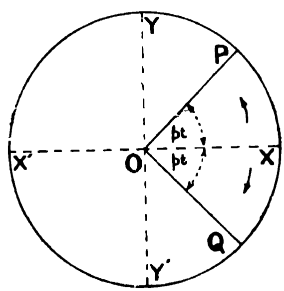


Fig. 8·33

If the rotor in single alternating field has an initial rotation both the rotating fields, as considered in the above, produce torque but in mutually opposite directions. The angular velocity of rotation relative to one field diminishes and relative to another increases, ultimately it becomes zero for one. The torque due to the other component of the field remains effective and the increase in velocity ceases when the mechanical work creates a balance. The coil as such runs as a motor rotating in the direction in which it is given a start. The direction of rotation can be reversed by reversing the conncetions to one coil.

In order to make the motor self-starting with a single-phase current an initial rotation of the armature is necessary. In order to obtain this one of the windings (Fig. 8·34) called the starting winding (S) is placed at half-a-pole pitch (90°) distance of the other (main) winding (M) and in parallel with it. The necessary phase difference is obtained between the currents in the two windings by any of the following methods. The motor consisting of arrangement of dividing the circuit in two parallel paths is known as a *split-phase motor.*

(i) **Capacitor starting :** A condenser in series (Fig. 8·34a) with the starting winding advances the phase of the circuit in this winding. The phase difference of 90° may not be possibly

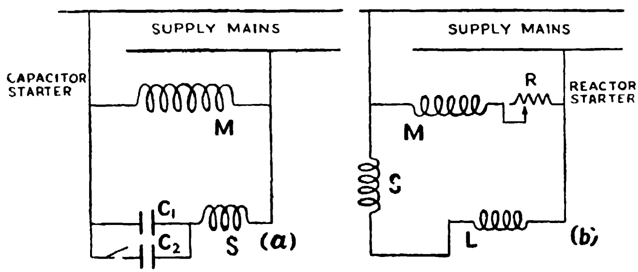


Fig. 8·34

obtained but the effect created enables the motor to start. The condenser may be cut out after the motor has gained speed or it may be left as it is, causing the motor virtually to work as two-phase machine. This method is greatly favoured now. For better performance the value of the capacitor when the motor is running should be different from the value when starting. For this purpose two condensers (C_1, C_2) are taken in parallel, one being cut out by a centrifugal switch when the motor has gained speed.

(ii) **Resistor starting :** A non-inductive resistance is placed in series with the starting winding. This tends to minimise the phase difference between the current and *emf* in this winding relative to that in the main winding. This causes the motor to stort. When the motor has gained speed, the starting winding is cut off by a centrifugal switch.

(iii) **Reactor starting :** A choke L (Fig. 8·34-b) in series with the starting winding (S) provides for the creation of a rotating field. When the motor has gained speed, the starting winding is cut off. The resistance R in series with the main winding is for limiting the current during starting and is cut out when the full speed is obtained. After the circuit of the starting winding has been cut off the motor runs on the main winding only.

POLY-PHASE MOTOR : Three-phase induction motor is more extensively used than in any other form of A.C. motor. The principle of working is based on the creation of a rotating magnetic field (§VIII-5) by means of three coils carrying current of three phases differing mutually by 120° and also placed at 120° apart from one another along the circumference of a circle. Three phase alternating current feeds the coils of such motors.

SLIP IN INDUCTION MOTOR : The couple acting on the rotor is proportional to the relative velocity of the rotating magnetic field at the rotor. Hence the motor must not run with the same speed as the field. Hence such a motor is called an *asynchronous motor.* Though the rotor would rotate theorctically at the same rate as that of the field there are reasons for this to be otherwise. At the synchronous speed the rotor current is zero and so the torque would vanish. Hence the actual speed becomes less. At no load the two speeds are nearly equal. When the load is applied the speed of the rotor falls to such a value at which the rotor current is just sufficient to provide for the increased torque necessary for the load. The *difference between the synchronous speed and the rotor speed is called slip.* It is usually expressed as a fraction of the synchronous speed. If p and p_o are the angular velocities of the field and rotor, then

$(p-p_o)$ is the absolute slip

$\frac{p-p_o}{p}$ is the fractional slip

and $\frac{p-p_o}{p}\times 100$ is the percentage slip

Synchronous motor : Just as a D.C. generator can be made to run as a motor, an alternator may also be worked in a similar way. Suppose that the stator (armature) core of an alternator is excited by an A.C. supply. The poles of a rotor (excited by D.C. and nature of poles remaining fixed) would be acted upon by a torque due to armature current. It will be in the same direction for all conductors. But at the end of each half-cycle, the armature current is reversed and the direction or torque also becomes opposite, so due to inertia the rotor would not move at all. Consider that the rotor has an initial speed which causes it move through one pole pitch in each half-cycle. Under this condition the current in each stator conductor is reversed when it finds the next opposite pole facing it. So the direction of the torque exerted by the armature conductors remains unchanged and the rotor continues to move. Hence the machine runs as a motor with contstant speed. Synchronous motor will have the same relationship between frequency (f), revolutions per second (n) and the number of poles (P) as the alternator, *i.e.* $f=nP$.

A synchronous motor working on a supply of fixed frequency will run at that particular speed at which it would have to be driven while functioning as an alternator to produce *emf* of the frequency supplied to it when working as a motor. A synchronous motor is of constant speed and requires start and synchronisation before it is loaded.

Universal motor : Series wound D.C. motor would run on A.C. as well. In such a motor the same current flows through the field and the armature coils. The torque produced is proportional to the square of the current and hence alternating current does not cause any disturbance regarding the motor action. But there would be undesirable effect due to eddy current and hysteresis. In order that the eddy current may be minimum the core of the field magnet must be laminated. Hysteresis effect must be low, otherwise there will be profuse heat production causing damage to the coil. For this special type of iron should be used.

VIII-7. POWER MEASURING DEVICES

Wattmeters : These are calibrated instruments for measuring the power consumed in an electrical circuit.

DYNAMOMETER-TYPE INSTRUMENT : It is essentially an electric motor containing a field coil (fixed) and a moving coil. The current (i) in the load circuit is fed into the field coil. The moving element which is directly connected to the supply voltage carries a current (i') proportional to the voltage e. The fixed coil sets up a magnetic field and the moving coil has a magnetic field along its own axis. The moving coil thus urged by torque tends to set its magnetic axis in line with that of the fixed coil. The moving system is controlled by a spring attached to it. The turning moment is proportional to the product of the current ($i.i'$) in the two coils. The restoring torque due to the spring is proportional to the deflection θ. In the deflected position the deflecting and the restoring torques are equal and hence

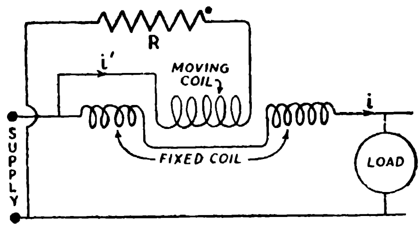


Fig. 8·35

$$\theta \propto i.i' \propto ei \propto P \text{ (Power consumed)}$$

The instrument may be used in A.C. and D.C. circuit. When used in A.C. circuit the torque is proportional to the product of voltage $e_0 sin\ \omega t$ and the current $i_0 sin\ (\omega t - \phi)$ where $cos\ \phi$ is the power factor.

$$\text{Torgue} \propto e_0 sin\ \omega t.\ i_0\ sin\ (\omega t - \phi)$$
$$\text{Average torque} \propto \tfrac{1}{2} e_0 i_0\ cos\ \phi \propto P$$

So in A.C. circuit also the deflection is proportional to the power.

The dial of the instrument is calibrated in power units and a pointer attached to the moving system indicates the power.

INDUCTION TYPE WATTMETER : It is an A.C. instrument. The current and voltage used in a circuit are

separately applied in the instrument to excite separate magnets creating separate fluxes. The flux produced links up with a light copper or aluminium disc. Eddy currents are produced in this disc separately by fluxes due to potential and current elements. Torque is produced by the action of flux on eddy current. There will be two sets of flux and two sets of eddy current. The flux due to potential element produces a torque on the eddy current due to current element and similarly the flux produced by current element exerts a torque on the eddy current due to potential element. These two torques cause a deflection of the disc which is a measure of the power absorbed in the A.C. circuit involved.

The applied *emf* excites the potential elements which is an electromagnet of shell-type made up of laminated core.

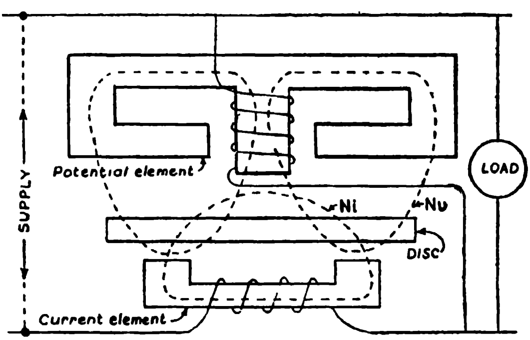


Fig. 8·36

The flux (N_v) produced is proportional to the voltage in the circuit and lags in phase be $\frac{\pi}{2}$ relative to the supply voltage. The magnetic circuit inside the shell is broken by a small air gap. There is thus considerable leakage flux which is embraced by the disc below it and causes an induced *emf* (e_v) in the disc, $\pi/2$ behind in phase with the flux which again is $\pi/2$ behind the supply voltage. Hence e_v is in opposite phase with E, the supply voltage. The eddy current i_v produced in the disc by e_v, *i.e.* by the potential element flux is in phase with e_v.

The current element produces a flux in a U-shaped magnet below the disc. The flux N_i is proportional to and in phase with the supply current (i). This flux produces an induced *emf* e_i and current i_i both in the same phase and $\pi/2$ behind the flux. The vector diagram (Fig. 8·37) shows the *emfs* and currents. The phase difference between supply voltage and load current is θ, which is also the phase between N_v and i_i. The phase difference between N_i and i_v is $(\pi-\theta)$.

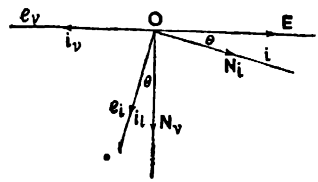


Fig. 8·37

The disc is subject to two torques as shown below. Torque due to interaction between N_v and i_i is

$$T_1 = N_v i_1 \cos\theta = K_1 Ei \cos\theta$$

Torque due to interaction between N_i and i_v is

$$T_2 = N_i i_v \cos(\pi-\theta) = K_2 Ei \cos\theta.$$

The resulting torque is the difference between the two and is proportional to average power $Ei \cos\theta$.

The instrument has a spiral control, the deflecting torque is equal to the restoring torque due to the spring. The shift of the zero position of the disc over a calibrated scale indicates the watts.

VIII-7. ENERGY METERS

Watt-hour Meter : The instrument is intended to measure total energy consumed in a circuit during interval. Wattmeters with an attached integrating device are used for this purpose.

ELIHU-THOMSON TYPE : It is an electric motor. The armature coil is connected to supply voltage. It has a high resistance (R) in series. A small current proportional to the supply voltage (E) flows through it. The field coil in series with the load circuit is fed by the load current (i). It has low resistance. When the load is applied the armature is subject to a field proportional to the load current. The torque T developed is proportional to the currents respectively in

the armature and field coils. The speed produced by the torque on the armature is controlled by the counter torque, generally obtained from a magnetic brake. It consists of a circular metal disc (copper or aluminium) attached to the shaft of the armature. The disc rotates through the poles of two C-shaped permanent magnets. Any radius of the disc during its rotation cuts the field of the magnet and there is induced eddy current in it. The eddy current reacts with inducing field and produces a drag upon the rotating disc. The disc and consequently the armature shaft both rotate with constant speed for a particular value of power consumed.

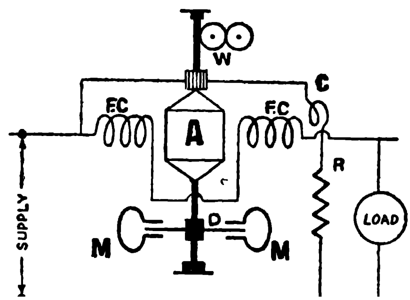


Fig. 8·38

$$\text{Torque } T \propto \frac{Ei}{R}$$

$$\text{Angular vetocity } \omega \propto \frac{Ei}{R}$$

If number of rotations in time t is n, then $n = \omega t$, hence

$$n \propto \frac{Eit}{R}$$

Since energy consumed in time t is $W = Eit$, so $n \propto W$.

To have a record of the total number of revolutions in an interval, the shaft is provided with a worm at the top which meshes with a reducing train of gears. The last gears are connected to pointers on the cyclometer type of dials calibrated in kilowatt-hours.

The meter may not run at all at very light loads because of the bearing friction. A coil (C) having a few turns of wire is connected in series with the armature coil and placed in a position parallel to the field coil. The coil produces a constant torque (irrespective of the load) arranged to be just

sufficient to overcome the friction. The load adjustment is made by shifting the permanent magnets towards or away from the centre.

INDUCTION-TYPE A.C. METER : An induction type wattmeter described is in a modified form converted into an energy meter. The control spring is replaced by brake magnets causing the disc to rotate at constant speed. It has been shown that the torque on the disc is proportional to the power and when it rotates due to a constant power supply at a constant speed, the number of revolutions preformed in an interval is proprotional to the enregy consumed in the period. The total number of revolution is recorded ay KWH units in an integrating device.

AMPERE-HOUR METER : In house supply where the voltage is maintained constant, simpler type of instruments are used to record (current × time). One common form of this ampere-hour meter is **Ferranti's mercury meter.**

A thin amalgamated copper disc (*C*) is mounted at its centre in jewelled cup bearings inside a shallow pan of a non-conducting material. The space inside the pan is filled with mercury. Two C-shaped magnets are placed on two sides of the disc with their poles one above and the other below the disc. The load current is led into mercury and therefrom it passes through a radius of the disc and the spindle and returns to the line. Any radius of the disc may be considered as a conductor lying under the poles of a magnet (driver magnet). The magnetic field produces a torque proportional to the current and causes the disc to rotate. During rotation the disc cuts through the field of (brake) magnet and induced eddy currents in the disc causes a counter torque. The speed of rotation increases till the deflecting and the restoring torques

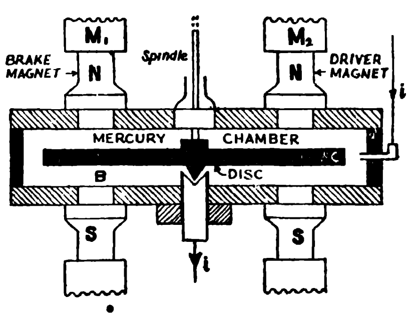


Fig. 8·39

are equal. Under such a condition the disc rotates with uniform speed proportional to the load current.

The spindle has gear wheels and recording dials attached with it. The number of revolutions in any interval is proportional to (current × time). If the *supply voltage is constant*, this number is also a measure of energy consumed. In calibrating the dial the number of revolutions are converted into Kilo-watt-hours by calculating the value of (volt × ampere × hours) corresponding to the observed value of number of revolutions, proportional to (ampere × hours) and assumed constant value of voltage.

It may be noted that this form of meter may be used with D.C. supply only. The record of the instrument will be somewhat above the actual consumption in KWH when the supply voltage falls to a value lower than that used while calibrating. So this form of meter will be a source of loss to consumer if the supply concern be not particular about maintaining the supply voltage at the scheduled maximum value.

VIII-8. FREQUENCY METER

Direct Reading Instrument : Frequeney of an A.C. supply may be obtained directly from the dial reading of a calibrated frequency meter. Inside the space bounded by two mutually perpendicular pairs of coil AA-BB a soft iron needle (N) is mounted on a spindle. The needle carries a pointer free to move over a calibrated scale. Each pair of coil has a resistance and a reactance in series. The inductance L_B of one pair and the resistance R_A of the other pair are in series across the supply. So any increase of potential drop

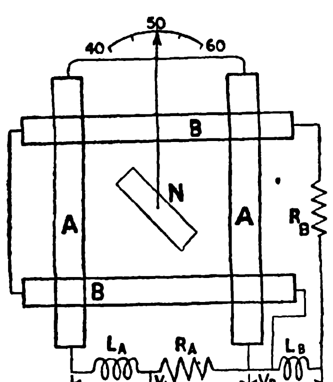


Fig. 8·40

in inductance in L_B causes in corresponding decrease in R_A

Consider that the pointer remains symmetrical on a scale for a definite frequency (say 50 c.p.s.) of the supply voltage. If the frequency increases the reactance of L_B will increase and this causes a rise of voltage across L_B and a fall of voltage across the resistance R_A. The rise of frequency also causes the reactance of L_A to increase. These two effects result in the rise of current in B and fall in A. As a result the needle would be deflected to come more in line with coils B-B. A decrease in frequency would show opposite effect. A calibrated scale would directly give the value of the changed frequency.

EXERCISES ON CHAPTER VIII

8·1. Obtain an expression for the *emf* generated in a straight conductor moving at right angles to a uniform magnetic field. How can this *emf* be made continuous? Hence explain the principle of a dynamo.

8·2. Describe the essential parts of a D.C. generator and explain with the help of Gramme-Ring armature the *emf* developed at the terminals of the machine. What are the disadvantages of this type of armature ?

8·3. Describe the drum-type armature of a generator. Indicate the differences between Lap and Wave windings. Show by diagrams the modes of winding in the two types.

8·4. Obtain the equation for the *emf* generated in a D.C. dynamo.

Calculate the speed at which the armature of a 6-pole D.C. dynamo having 664 wave-wound conductors in a field of 6×10^8 maxwells must be rotated to generate an *emf* of 500 volts. [*Ans* : 250 r.p.m.]

8·5. What is armature reaction in a dynamo ? What should be done to combat with this ?

Describe different ways of combining the field and the armature coils in a D.C. generator and discuss the characteristic of each form.

8·6. Describe the essential parts of an A.C. generator and explain the difference between single phase, two phase and polyphase machines.

Obtain the equation for the *emf* generated.

8·7. Obtain the expression for the frequency of the *emf* generated by an alternator in terms of number of poles and revolutions per second.

Calculate the frequency of the *emf* generated in a 12-pole generator if it rotates at 500 revolutions per minute.

[*Ans* : 50 c.p.s.]

8·8. Draw diagrams to show the distributions of *emf* from three-phase star-wound and delta-wound alternators.

What are meant by phase-voltage and line-voltage ? Obtain the relations between them.

8·9. Describe the principle of a D.C. motor and obtain an equation the torque acting on the armature. What is back *emf* in a motor ?

'Back-*emf* makes the motor a self-adjusting machine'—explain.

8·10. Explain the difference between a series and a shunt motor.

Why should a motor require a starting resistance ? Explain the working of the motor starter attached to (a) series motor, (b) shunt motor.

8·11. What is a rotating magnetic field ? Discuss its productions by (i) two coils, (ii) three coils.

Obtain an expression for the torque acting on a coil placed in a rotating magnetic field. Hence give the outlines of construction of an induction motor.

8·12. Describe the principle of an induction motor. What is *slip* in such a motor ?

Explain how an induction motor can be run with a single-phase current.

8·13. Describe and explain the working of a wattmeter.

8·14. (a) Indicate the working of an energy meter. (b) Describe an induction type energy meter.

8·15. Write notes on :

(a) Synchronous motor
(b) Ampere-hour meter
(c) Frequency meter
(d) Split-phase motor
(e) Universal motor

CHAPTER IX

UNITS AND DIMENSIONS

IX-1. DIMENSIONS

Expressing a physical quantity in fundamental units : Units of mass, length and time are known as fundamental units. Other physical quantities are measured in various separate units. But whatever may be the principle of fixing such standards or units of measurement, each of these can be expressed in terms of the products of fundamental units raised to various powers. The power to which any fundamental unit is to be raised in expressing any other physical quantity is called its dimension. As for example volume may be expressed as products of three lengths. So unit of volume is of the third dimension of unit of length. Similarly, since velocity is measured as length/time, if expressed in dimensions of fundamental units unit of velocity should be shown as $[LT^{-1}]$. For the same reasons, acceleration has the dimensions $[LT^{-2}]$. Force measured as mass$\times$acceleration is written in dimensional equation as $[F]=[MLT^{-2}]$. Any other physical quantity may be expressed in a similar equation involving M, L and T.

Electrical units : Units for measurement of different electrical quantities have been defined from various electrical properties. Whatever may be the considerations in defining any such unit, it is possible to express it in dimensions of fundamental units. There are two systems of defining an electrical unit, electromagnetic and electrostatic. Different unit in the two systems have been obtained from different properties of static and moving charges. Whatever may be basis of defining a particular unit, expressed in fundamental units a quantity must have the same dimensions in both the systems. The study of such similarity reveals some important facts about the nature of electricity.

Dimensions of different electrical units may be obtained in consideration of the physical relation between them. These may be developed in a logical sequence from the fundamental property of moving or static charge, the force action. These are shown in the following sections.

IX-2. DIMENSIONS OF ELECTRO-MAGNETIC UNITS.

Strength of magnetic pole : The interaction between two magnetic poles is in the nature of a force. Force acting between two poles of strength m and m' separated by a distance r, in a medium of permeability μ is calculated from the formula deduced from experimental results and written as

$$F = \frac{mm'}{\mu r^2}$$

In this equation μ is a quantity whose value in vacuum is taken to be 1 to have a convenient unit of measurement of pole strength. According to this convention force between two unit poles ($m = m' = 1$) separated by unit distance ($r = 1\ cm$) becomes one dyne in air. Though according to this assumption μ is a numerical constant (for air $\mu = 1$ for practical purpose) it is really not so. It is in fact, a quantity with dimensions, which, however cannot be ascertained.

Considering the equality of dimensions of fundamental units involved on both sides of the aforesaid equation, we may write,

$$\left[MLT^{-2}\right] = \left[\frac{m^2}{\mu L^2}\right]$$

So dimensions of m is obtained as

$$\left[m\right] = \left[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} \mu^{\frac{1}{2}}\right]$$

Magnetic field strength : To make an estimate of field strength it has been assumed that a pole of strength m experiences a force mH in a field of intensity H. Hence according to this definition, the product mH is a force in dimensions, so

$$[F] = [mH]$$

Considering dimensions on both sides,

$$[MLT^{-2}] = [M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}H]$$

$$\text{or} \qquad [H] = [M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$$

Magnetic Induction : Using $B=\mu H$

$$[B]=[\mu H]=[\mu^{\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}]$$

Magnetic Flux : Using Flux (N)=Induction $(B)\times$Area

$$[N] = [\mu^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}]$$

Electric Current : The strength of current flowing through a conductor is estimated by its magnetic effect. The field (H) due to a conductor of length δl in a direction θ at a distance r when a current i flows through it is measured as

$$H = \frac{i.\delta l.\sin\theta}{r^2}$$

$$\text{or} \quad i = \frac{H.r^2}{\delta l.\sin\theta}.$$

Considering dimensions, $[\,i\,] = \dfrac{[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}L^2]}{[\,L\,]}$

$\sin\theta$ is a ratio and has no dimensions, hence

$$[\,i\,] = [M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$$

Electric charge : The quantity of charge $(\,Q\,)$ carried out by a current i in time t is according to definition of unit charge is

$$Q = i.t$$

$$\text{Hence } [\,Q\,] = [M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}T] = [M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}]$$

Electromotive force or Potential difference : If an *emf* transfers a charge Q between two points differing in potential by V, the work done is given by

$$W = VQ$$

$$\text{or} \quad V = \frac{W}{Q}$$

$$\text{So} \quad [V] = \frac{[ML^2T^{-2}]}{\left[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}\right]}\left[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}\right]$$

The dimensions of *emf* and potential are same.

Resistance : According to Ohm's Law, the resistance (R) is related to potential difference (V) and current (i) in the form expressed as

$$R = \frac{V}{i}$$

$$\text{or} \quad [R] = \frac{\left[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}\right]}{\left[M^{\frac{1}{2}}L^{\frac{1}{2}}L^{-1}\mu^{-\frac{1}{2}}\right]} = [LT^{-1}\mu]$$

Inductance : Induced *emf* (e) in an inductance (l) is measured as

$$e = -l\frac{di}{dt}$$

$$\text{or} \quad l = -e/\frac{di}{dt}.$$

$$\text{So} \quad [l] = \frac{\left[M^{\frac{1}{4}}L^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}\right]}{\left[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu\right]} = [L\mu]$$

Capacitance : Capacitance of a charged conductor is calculated as the ratio of its charge (Q) and potential (V)

$$C = \frac{Q}{V}$$

$$[C] = \frac{\left[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}\right]}{\left[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{2}\mu^{\frac{1}{2}}\right]} = [L^{-1}T^{2}\mu^{-1}]$$

Electric intensity or field strength : The force on a charge (Q) in a field of intensity E is $F=EQ$, hence

$$(F)=[EQ]$$

$$\text{or} \quad [MLT^{-2}]=[E]\left[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}\right]$$

$$\text{so} \quad [E]=\left[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}\right]$$

IX-3. DIMENSIONS OF ELECTRO-STATIC UNITS

Electric charge : Force between two charged bodies forms the basis of fixing electrostatic units of measurement. Two charges Q and Q' separated by a distance r, in a medium of dielectric constant K experiences a force measured as

$$F=\frac{QQ'}{Kr^2}$$

$$\text{or} \quad [Q^2]=[Fr^2K]$$

$$\text{or} \quad [Q]=[MLT^{-2}KL^2]^{\frac{1}{2}}=\left[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}\right]$$

K though considered as a numerical constant, it is not so. It is a quantity having dimensions, which however cannot be ascertained.

Potential difference : The work (W) involved in the transference of charge (Q) between two points having a potential difference (V) is measured as

$$W = QV$$

$$\text{or} \quad V=\frac{W}{Q}$$

$$\text{or} \quad [V]=\frac{[ML^2T^{-2}]}{\left[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}\right]}=\left[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}\right]$$

Electric Field : Using, electric intensity $E=-\frac{dV}{dx}$

$$[E]=[V]\,[L^{-1}]=\left[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}\right]$$

Electric Induction : Using, electric induction $D=KE$

$$[D]=\left[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}K^{\frac{1}{2}}\right]$$

Electric current : Current (i) is measured as the rate of flow of charge (Q), so

$$i=\frac{Q}{t}$$

$$\text{or} \quad [i]=\frac{\left[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}\right]}{[T]}=\left[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}}\right]$$

Capacitance : Capacitance (C) is measured as the charge (Q) required by a capacitor to raise its potential (V) by unity, hence

$$C=\frac{Q}{V}$$

$$\text{or} \quad [C]=\frac{\left[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}\right]}{\left[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}\right]}=[LK]$$

Resistance : Resistance according to Ohm's law is the ratio of the potential difference (V) and the current (i).

$$R=\frac{V}{i}$$

$$\text{or} \quad [R]=\frac{\left[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}\right]}{\left[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}K^{\frac{1}{2}}\right]}=[L^{-1}TK^{-1}]$$

Inductance : If l be the inductance, energy (W) stored in a coil carrying a current i is given by

$$W=\tfrac{1}{2}li^2$$

$$\text{or} \quad [ML^2T^{-2}]=[l]\,[ML^3T^{-4}K]$$

$$\text{or} \quad [l]=[L^{-1}T^2K^{-1}].$$

IX-4. RATIO OF TWO UNITS

Dimensions of $K\mu$: It may be observed that as expressed in dimensions of mass, length and time, the same electrical quantities in the two systems of units are found to have dissimilar dimensions. It seems unreasonable. This apparent discrepancy lies in the fact that the dimension of μ and K have not been considered. As the quantites μ and K have been taken as constant numbers in fixing up certain units, dimensions of μ and K cannot be obtained by any logical deduction. By equating the dimensions of same electrical quantity in the two systems we can obtain the dimensions of ($K\mu$) taken together.

Let the capacitance of a capacitor measured in *e.s.* unit and *e.m.* unit respectively by C_e and C_m.

Considering that since they represent the same quantity their dimensions must be same, so we may write,

$$C_e[LK] = C_m[L^{-1}T^2\mu^{-1}]$$

$$\text{or } \left[\frac{1}{K\mu}\right] = \frac{C_e}{C_m}[L^2T^{-2}]$$

$$\text{or } \frac{1}{[\sqrt{K\mu}]} = \sqrt{\frac{C_e}{C_m}}[LT^{-1}]$$

The dimensions of $\frac{1}{\sqrt{K\mu}}$ is that of velocity and its magnitude (for the same medium) is given by $\sqrt{C_e/C_m}$. This numerical quantity can be obtained by measuring the capacitance of the same capacitor in two units. By experimental determination of the capacitance of an air condenser in the two systems of units separately the magnitude of $1/\sqrt{K\mu}$ is found to be equal to 3×10^{10}. Since this has the dimensions the value of $1/\sqrt{K\mu}$ for air or vacuum is found to be 3×10^{10}centimetres per second. This is also the velocity of light in vacuum. So denoting c as the velocity of light in vacuum and K_0, μ_0 as the permittivity and permeability of vacuum respectively, ***measured in the same system of units***, we may write

$$\frac{1}{\sqrt{K_0\mu_0}} = c$$

As a quantity without dimensions c would mean a ratio having a numerical value 3×10^{10}.

Likewise the ratio of other electrical quantities measured in the two systems may be obtained as shown below

$$\frac{i_m}{i_e} = c,\ \frac{R_m}{R_e} = c^{-2},\ \frac{V_m}{V_e} = c^{-1},\ \frac{Q_m}{Q_e} = c,\ \frac{l_m}{l_e} = c_2.$$

Maxwell's conceptions : The fact that $1/\sqrt{K_0\mu_0}$ is a quantity having its value in magnitude and dimensions equal to the velocity of light in vacuum led Maxwell to the idea that there is some connection between the propagation of light and electricity. Hence he conceived the postulate that light waves

are electromagnetic in nature. He eventually established the fact by means of mathematical equation (see Chapter X) relating to the electric and magnetic conditions of space when it propagates light waves or more generally bringing in the conception of electro-magnetic waves. Thus the idea of material light waves propagating through the hypothetical medium ether could be discarded.

Experimental determination of Ratio of units: The ratio of the dimensions of any electrical quantity in the electrostatic and electromagnetic units is found to be dimensions of velocity itself or velocity raised to a power, positive or negative, integral or fractional. The magnitude of the ratio can be obtained by measuring the same electrical quantity in the two systems of units. For this purpose the capacitance of a condenser is measured by two separate methods involving electrostatic and electromagnetic units.

A condenser is made by fixing layers of tin foils upon two sheets of glass. One of the tin foils is circular having a guard-ring. The two glass plates are placed with metallic surfaces lying face to face separated by air. The plates are held in position by ebonite separators (Fig. 9·1).

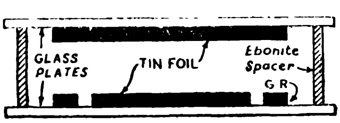


Fig. 9·1

CAPACITANCE IN E. S. UNITS: The capacitance C_e of a parallel plate air condenser is obtained from the formula

$$C_e = \frac{A}{4\pi d},$$

where A is the area of the circular plate and d is the distance separating the two plates.

CAPACITANCE IN E. M. UNITS: In electromagnetic unit the capacitance may be measured either by a ballistic galvanometer or by Maxwell's method.

(a) *Ballistic galvanometer method*: The circular plate is charged to a high potential (V) and condenser is discharged

through a ballistiic galvanometer. If θ be the observed throw, λ the log-decrement, C_m the capacitance and Q the charge, then

$$C_m . V = Q = \frac{\alpha T}{2\pi AH} \theta \left(1 + \frac{\lambda}{2}\right) \quad \ldots \quad \text{(i)}$$

The galvanometer constant α/AH is eliminated by obtaining a steady deflection θ_0 for a known current produced by applying a small potential V_1 across a high resistance R so as to cause a current $i = V_1/R$ to flow through it. For the deflection θ_0,

$$iAH = \alpha\theta_0$$

$$\text{or} \quad \frac{V_1}{R} AH = \alpha\theta_0$$

$$\text{or} \quad \frac{\alpha}{AH} = \frac{V_1}{R\theta_0} \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (i) and (ii)

$$C_m = \frac{V_1 T}{2\pi VR} \cdot \frac{\theta}{\theta_0}\left(1 + \frac{\lambda}{2}\right)$$

T, the period of the suspended system is to be obtained by direct observation. C_m is calculated from the foregoing equation.

(b) **Maxwell's method** : If a condenser be placed in series with a battery and a galvanometer it will receive a charge $e.C_m$, where e is the *emf* of the battery and C_m is the capacitance of the condenser in *e.m.* units. As soon as the circuit is closed the galvanometer shows a throw but there is no steady deflection. If the condenser is short-circuited it is discharged.

Let an arrangement be made so that the galvanometer may be charged and discharged in quick succession, say n-times per second. In such a case the charge flowing per second is neC_m, which is equivalent to a steady current of same magnitude. If the frequency n be very high and the galvanometer has a long period of oscillation compared with $1/n$, the intermittent currents through the galvanometer will produce a steady deflection.

Such a rapid charge and discharge may be arranged by an electrically maintained tuning fork of known frequency with a rocker connected with the prong or by a revolving commutator system. The electric connections are as shown in the diagram (Fig. 9·2). When the rocker (D) is in contact with a, the condenser gets a charge and it is discharged when the rocker moves upto b. Again in the same circuit the condenser with the vibrator arrangement is replaced by a variable resistor r in the circuit. Generally r will be very high, so the galvanometer and the battery resistances may be neglected in calculation of current. r is so adjusted that the deflection becomes equal to that shown during charge and discharge. So the current through the galvanometer may be obtained as

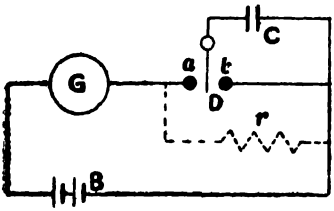


Fig. 9·2

$$i = \frac{e}{r} = neC_m$$

$$\text{or} \quad C_m = \frac{1}{nr}$$

Hence from the two experiments giving the values of C_e and C_m, the value of $\sqrt{C_e/C_m}$ may be calculated and value of C obtained.

This method of determining the ratio of units is also an indirect method of determining the velocity of light.

The experiment described shows that for a rapid charge and discharge a condenser included in a battery circuit behaves as a resistance. The equivalent resistance of a condenser in such a case may also be obtained, instead of substitution method, by a wheatstone bridge arrangement as described overleaf.

Maxwell's bridge for determining the equivalent resistance of a capacitor : The condenser with the vibrator arrangement is placed in the fourth arm of a wheatstone bridge. The arrangement is shown in the diagram (Fig. 9·3). For a balance in the bridge the potentials at B and O are same, *i.e.*

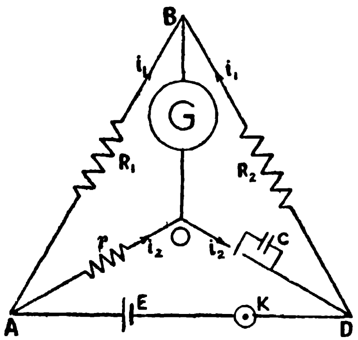


Fig. 9·3

$$V_B = V_o$$

or $i_1 R_1 = i_2 r$ ... (i)

and $i_1 R_2 = V$, the potential of the condenser ... (ii)

But $i_2 = nVC_m$ ... ... (iii)

By substitution in (*i*), the value of i_1 and i_2 as obtained from (ii) and (iii), we get

$$nVC_m r = V.\frac{R_1}{R_2}$$

$$\text{Hence} \quad C_m = \frac{1}{nr} \cdot \frac{R_1}{R_2}.$$

ILLUSTRATIVE EXAMPLES

1. Calculate how many electromagnetic unit of capacitance is equal to 1 electrostatic unit of capacitance.

Solution : Energy of a charged condenser $W = \frac{1}{2}.\frac{Q^2}{C}$ In electrostatic system a condenser having 1 e.s.u. of capacitance and 1 e.s.u. of charge has an energy given by

$$W = \frac{1}{2} \cdot \frac{1^2}{1} = \frac{1}{2} \text{ erg}$$

The energy of the same condenser with same charge the magnitude being expressed in e.m.u. will be the same in ergs.

1 e.s.u. of charge $= \frac{1}{3 \times 10^{10}}$ e.m.u. of charge

Hence $W=\frac{1}{2}\left(\frac{1}{3}\times 10^{-10}\right)^2 \cdot \frac{1}{C_m}$ ergs.

Equating the two the expressions

$$\frac{1}{2\times 9}\times 10^{-20}=\frac{1}{2}C_m$$

$$\text{or} \quad C_m=1.11\times 10^{-19}$$

Alternatively,

$$C_e[LK_o]=C_m[L^{-1}T^2\mu_o^{-1}]$$

$$\frac{C_e}{C_m}\left[L^2\,T^{-2}\right]=\frac{1}{K_o\mu_o}$$

$$\text{So} \quad \frac{1}{C_m}=(3\times 10^{10})^2 \cdot \frac{1}{C_e}=\frac{9\times 10^{20}}{C_e}$$

$$\text{or} \quad C_m=\frac{1}{9\times 10^{20}}=1{\cdot}11\times 10^{-19}$$

2. *Express 10^6 e.m. units of inductance in e.s. units.*

Solution : $i_m[L\mu_o]=i_e[L^{-1}T^2K_o^{-1}]$

$$\frac{i_m}{i_e}[L^2T^{-2}]=\frac{1}{K_o\mu_o}$$

$$\text{or} \quad \frac{i_m}{i_e}=C^2=(3\times 10^{10})^2=9\times 10^{20}$$

$$\text{or} \quad i_e=\frac{i_m}{9\times 10^{20}}=\frac{10^6}{9\times 10^{20}}$$

$$\text{or} \quad i_e=1{\cdot}11\times 10^{-13} \text{ e.s.u.}$$

3. *Calculate $1\mu F$ capacitance in e.s. unit.*

Solution : Let $1\mu F$ capacitor be charged with a p.d. of 1 volt.

$$\text{Energy}=\tfrac{1}{2}CV^2=\tfrac{1}{2}\times 10^{-6}\times 1^2 \text{ joule}$$

$$=\tfrac{1}{2}\times 10^{-6}\times 10^7=5 \text{ ergs.}$$

Let $1\mu F$ be equivalent to x *e.s.* units of capacitance, its energy when a p.d. of 1 volt, *i.e.* $\frac{1}{300}$ e.s. units is applied will be also 5 ergs.

$$\text{So} \quad \frac{1}{2}\,x\times\frac{1}{(300)^2}=5 \text{ ergs}$$

$$\text{or} \quad x=9\times 10^5.$$

IX-5. ABSOLUTE DETERMINATION OF UNITS

Construction of standards : The electromagnetic unit of current has been defined from the force action on a magnet. Galvanometer is a current measuring device. But it cannot be used to give an absolute measure of current based on definition. This is because of the difficulties and uncertainties involved in determination of constants. Ammeters and voltmeters are calibrated instruments. But calibration requires comparison with measuring devices giving absolute values as demanded by the definition of units. Further, only current measurement is not sufficient for determination of all electrical quantities. Ohm's law corelates current, electromotive force and resistance. If methods are devised for absolute measurement of two of these, the third may be determined by application of Ohm's law. For this purpose, we have got methods for absolute measurements of current and resistance.

IX-6. DETERMINATION OF OHM

Rotating Coil method : This is the method adopted by British Association in 1863 for constructing a standard of resistance.

A closed coil of radius a having n turns each of area A has a total resistance R. The coil is capable of rotation about a vertical axis at a high speed (angular velocity ω). A small magnet free to rotate, is kept at the centre of of the coil. When the coil is rotated, the field due to the induced current in it causes a deflection of the magnet by an angle (say θ) from the magnetic meridian.

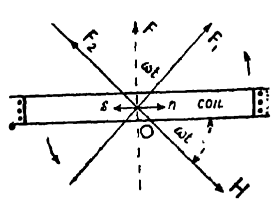


Fig. 9·4

At any instant, the magnetic needle is acted upon by

(*i*) the magnetic field (F) at the centre of the coil due to induced current. F acts in the direction of the normal to the plane of the coil.

(*ii*) the earth's horizontal magnetic field (H_0). The induced *emf* in the coil is given by $e = -\frac{dN}{dt}$. The normal flux (N) through the coil due to earth's field at any instant when the plane of the coil makes an angle ωt with the direction of H_0 is $nAH_0 \sin \omega t$.

$$\text{Hence } e = -\frac{dN}{dt} = -\frac{d}{dt}(nAH_0 \sin \omega t) = -AnH_0\omega \cos \omega t$$

$$\text{and the induced current } i = \frac{nAH_0\omega}{\sqrt{R^2+L^2\omega^2}} \cos(\omega t - \phi)$$

L is the self-inductance of the coil and $\tan \phi = \frac{L\omega}{R}$.

The magnetic field (F) due to the current at the centre of the coil is $F = 2\pi ni/a$, acting at right angles to the plane of the coil. Its component F_1 at right angles to the magnetic meridian, and the component F_2 acting in the same direction as the magnetic meridian are obtained as shown below.

$$F_1 = \frac{2\pi ni}{a} \cos \omega t = \frac{2\pi n}{a} \cdot \frac{nAH_0\omega}{\sqrt{R^2+L^2\omega^2}} \cos(\omega t - \phi) \cos \omega t$$

Its average value over a complete cycle is

$$\overline{F_1} = \frac{\pi n^2 AH_0\omega}{a\sqrt{R^2+L^2\omega^2}} \cos \phi = \frac{\pi^2 n^2\, a\, H_0\omega}{\sqrt{R^2+L^2\omega^2}} \cos\phi$$

$$\text{Again } F_2 = \frac{2\pi ni}{a} \sin \omega t = \frac{2\pi n}{a} \cdot \frac{nAH_0\omega}{\sqrt{R^2+L^2\omega^2}} \cos(\omega t - \phi) \sin \omega t$$

Average of F_2 over a complete cycle is

$$\overline{F_2} = \frac{\pi n^2 AH_0\omega}{a\sqrt{R^2+L^2\omega^2}} \sin \phi = \frac{\pi^2 n^2\, a\, H_0\omega}{\sqrt{R^2+L^2\omega^2}} \sin \phi$$

So deflection θ of the magnet considering its equilibrium in mutually perpendicular fields is obtained as

$$\tan\theta = \frac{\overline{F_1}}{H_0 - \overline{F_2}} = \frac{\pi^2 n^2 a H_0\omega \cos \phi / \sqrt{R^2+L^2\omega^2}}{H_0 - \frac{\pi^2 n^2 a H_0\omega}{\sqrt{R^2+L^2\omega^2}} \sin \phi}$$

or $$\tan\theta = \frac{\pi^2 n^2 a\omega \cos\theta}{\sqrt{R^2+L^2\omega^2}-\pi^2 n^2 a\omega \sin\phi}$$

This equation gives the value of R in *e.m.* units in terms of θ, ϕ, L, ω, n and a; the magnitudes of these may be obtained directly from experimental procedure and observations.

If L is so small that $\frac{L\omega}{R} \rightarrow 0$, then $\cos\phi \rightarrow 1$ and $\sin\phi \rightarrow 0$, in such a simplified case

$$\tan\theta = \frac{\pi^2 n^2 a\omega}{R}$$

$$\text{or} \quad R = \pi^2 n^2 a\omega \cot\theta.$$

The effect of torsion in the suspension fibre and of the field due to the suspended magnet itself are the corrections to be applied for accurate determination.

Lorenz's method : A copper disc (D) is placed inside a coil (A) carrying a current from a battery. When the disc is rotated about horizontal axis an *emf* is generated between the centre and the circumference of the disc due to the fact that during rotation any radius of the disc cuts the flux inside the coil produced by the current flowing through the coil.

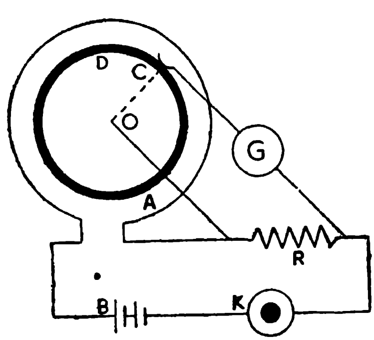


Fig. 9·5

To avoid generation of *emf* due to rotation in the earth's field the disc is placed in the magnetic meridian. The induced *emf* at the ends of any radius (say OC) of the disc may be balanced against the potential drop across any suitable resistance (R) placed in series with the coil. The balance may be observed in a sensitive galvanometer (G) included in the circuit. The null in the galvanometer is obtained by adjusting the speed of rotation of the disc.

Let i be the current flowing through the coil and the resistor. If M is the mutual inductance between the coil and the disc and n is the number of revolutions per second of the disc, then the induced *emf* at the ends of OC is nmi. The potential drop at the ends of R is Ri Hence when a balance is obtained, we have

$$Ri = nMi$$

$$\text{or} \quad R = nM.$$

M is calculated by applying a standard formula concerning dimensions of the coil and the disc.

EXPERIMENTAL ARRANGEMENT : In actual expriment two coaxial coils are used and the disc is placed inside the coils in a symmetrical field. To counter balance the *emf* that may be produced (*i*) due to thermal effects and (*ii*) for the disc being not exactly in magnetic meridian a suitable *emf* is introduced in the circuit through the ends of a resistor R_2. When the coil carries no current, the disc is made to rotate. If there be any deflection due to the *emf* for reasons stated in (*i*) and (*ii*) potential drop at the ends of R_2 which is in circuit with the galvanometer is adjusted to obtain a null. The motion of the disc is produced by an electric motor coupled through a pulley (*P*). The speed is adjusted after passing a current through the coil till the balance is obtained.

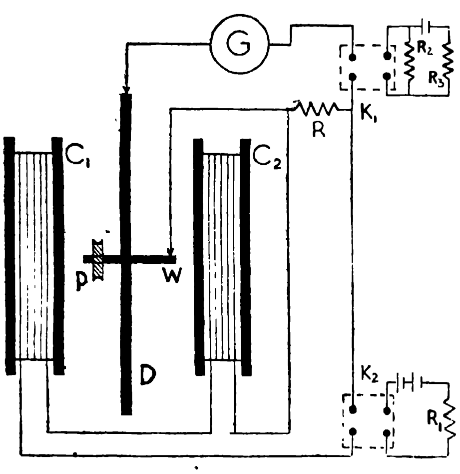


Fig. 9·6

Modified arrangement : To eliminate the thermal effects and the induced *emf* due to earth's field the arrangement has

been modified in which a double system of coils and discs are used. The two discs are on the same shaft and are rotated in the same direction. The magnetic fields produced inside the coils are in mutually opposite directions. By arranging suitably the contacts at the brushes (as shown in the sketch) the induced *emfs* in the disc due to the coils are made additive, but *emfs* due to thermal effects and earth's field in the two remain in opposition. The shaft is made of copper-aluminium to avoid the effect of any magnetic material near the disc. The motor is kept at considerable distance from the discs.

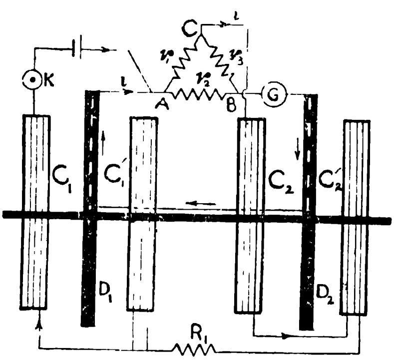


Fig 9·7

Assuming that the discs and the coils are identical, in this arrangement we have for null $R=2nM$. So for a balance the speed of rotation may be in this case reduced to half its value with a single coil. This reduces the heating effect. The dimensions and relative positions of the coils are so chosen that at the rims the flux-density due to current in the coils is minimum. The effect of small uncertainties in the estimation of the radius of disc is thus minimised and mutual inductance becomes maximum.

Lord Rayleigh and Mrs. Sidgwick used an arrangement in which the resistance R instead of being a single low resistance was constructed of three resistances in parallel. r_1 is the smallest resistance, while r_3 is large compared with r_2. If a current i is supplied is to to the combination by corrections as shown in the diagram (Fig. 9·8), the greater portion of current flows through r_1. The potential difference between A-B is exceeding small. The

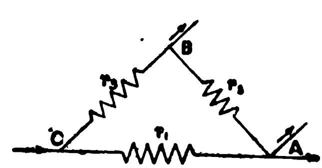


Fig. 9·8

current through r_2 and r_3 is $\frac{r_1 i}{r_1+r_2+r_3}$ and the potential between across AB is $\frac{r_1 r_2 i}{r_1+r_2+r_3}$. The arrangement provides for an equivalent resistance $\frac{r_1 r_2}{r_1+r_2+r_3}$ across AB, which takes place of R in the equation $R=nM$. Thus R becomes very small as is necessary for the experiment. In connecting this combination r_1 is in series with the coils and r_2 is in series with the galvanometer circuit.

SECONDARY STANDARD which can be readily reproduced has been made from absolute determination. **International ohm** is the resistance of a column of pure mercury of uniform cross section, 14·4521 grammes in mass and of length 106·30 centimetres at 0°C.

IX-7. ABSOLUTE MEASUREMENT OF CURRENT

Rayleigh's current balance method : The absolute measurement of current based on the force action between two coils

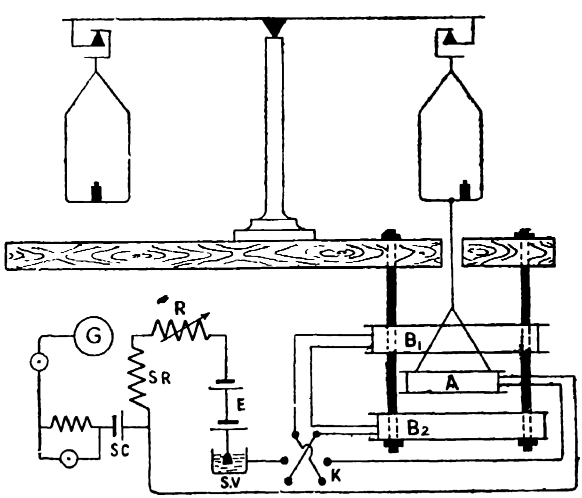


Fig. 9·9

carrying current. Such a force is balanced against gravitational force on a body of known mass.

The experimental arrangement for determining the force action between current elements, consists of three coils of square cross-sections. Two larger coils are fixed (B_1, B_2 in Fig. 9·9). The smaller coil (A) is placed between the larger coils and it is suspended from the pan of a suitable precision balance and it lies in a horizontal plane. The coils carry the same current. The currents in the larger coils are so directed that the smaller coil experiences forces in the same direction due to both coils, and the effective force is double of that due to a single coil.

The suspended coil is first counterpoised in the balance when there is no current. A current is passed so as to produce a force causing downward movement of the suspended coil. The counterprising weight is adjusted to bring the balance again in equilibrium. The current in the fixed coil is then reversed in direction so as to cause an upward drag on the movable coil. A standard mass m is now placed on the pan from which the coil has been hung to restore equilibrium.

If M is the mutual inductance of the coil system carrying current i, the mutual potential energy due to mutual inductance is given by $W = Mi^2$ and the mutual force is obtained as

$$F = \frac{dW}{dx} = \frac{d}{dx}(Mi^2) = i^2 \frac{dM}{dx}$$

so $$2i^2 \frac{dM}{dx} = mg$$

The value of $\frac{dM}{dx}$ is calculated from the dimensions of the coils and the distance separating them, with necessary corrections for errors in measurement of linear dimensions. i is determined from the equation in the above.

If a silver voltameter is connected in series with the coil current, the mass of silver deposited by one ampere current flowing for one second may be obtained by experimental observation. This forms a *secondary standard* for current measurement. The **International Ampere** is the current that causes a deposit of silver at the rate of $1{\cdot}1180 \times 10^{-3}$ gramme per second. Further work has shown that 0·1 e.m. unit of

current deposits silver at the rate of $1{\cdot}11805 \times 10^{-3}$ gramme per second. This amount of current is called *absolute ampere.*

Determination of volt : The current through the coil may be utilised to produce a balance in a circuit containing a standard cell and a standard resistance. Hence the value of the cell mny be expressed in terms of the resistance and current in the circuit.

The volt may be defined from the *emf* of a standard cell thus determined. It is $\frac{1}{1{\cdot}0183}$ part of the potential difference between the plates of a Weston Cadmium cell at 20°*C*. The *emf* of the cell is taken as 1·0183 volts at 20°*C*.

EXERCISES ON CHAPTER IX

9 1. Find from first principles the dimensions of resistance in electromagnetic and electrostatic systems.

Discuss why the dimensions differ in the two systems. What is the significance of the ratio of two units ? Describe a method of obtaining the ratio experimentally.

9·2. What led Maxwell to the conception of electromagnetic nature of light waves ?

Describe the indirect method of obtaining the velocity of light by electrical method.

9·3. Describe the rotating coil method of determining the value of ohm in absolute measure.

A closed coil of 30 turns of wire and 50 cm. radius is rotated uniformly about a vertical axis passing through the plane of the coil at a speed of 1800 revolutions per minute. If the rotation of the coil deflects a magnetic needle placed at the centre of the coil through 45° from its rest position, calculate the resistance of the coil. (Neglect the self-inductance of the coil). [*Ans.* 1321×10^4 ohms].

9·4. Describe the rotating disc method of determining the ohm in absolute measure. State the conditions for obtaining an accurate result.

CHAPTER X

ELECTRO-MAGNETIC RADIATION

X-1. MAXWELL'S FIELD EQUATIONS

Fundamental field properties : Maxwell mathematically deduced an equation in which the quantity $1/\sqrt{K\mu}$ is obtained as a velocity, K and μ being respectively the permittivity and permeability of a medium. Maxwell's equations leading to this evaluation are concerned with inter-relationships between electric intensity (E) and the magnetic force (H). Different sets of equations are obtained from different laws regarding electrostatic and magnetic fields. These are deduced as shown in the following sections.

EQUATIONS RELATING TO GAUSS'S THEOREM : This theorem in electrostatics connects the total normal induction (D) over a closed surface and the charge inside the enclosed volume. Conveniently taking this volume as a parallelopiped bounded by sides of length δx, δy, δz, the theorem is expressed as

$$\iint D.dS = 4\pi\rho.\delta x.\delta y.\delta z,$$

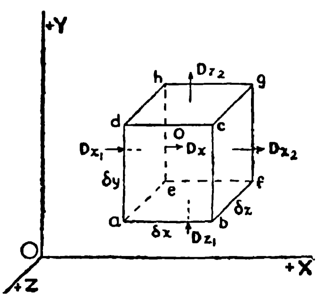


Fig. 10·1

where ρ is the volume density of charge. The quantity $\iint D.dS$ over the surface of a the parallelopiped (Fig. 10·1) is obtained as the sum of six contributions, one from each face. This again is equal to the area of the face multiplied by the mean value of induction at the centre of the face. Considering D_x to be the induction at the centre of the parallelopiped in the direction of x, and taking into account the direction in which D_x acts over the faces, the algebraic sum

of the contributions from the faces *bfgc*, *aehd* may be obtained as

$$\left(D_x+\frac{\partial D_x}{\partial x}.\frac{\delta x}{2}\right)\delta y.\delta z-\left(D_x-\frac{\partial D_x}{\partial x}.\frac{\delta x}{2}\right)\delta y.\delta z$$

Similarly the contributions from the other two pairs of faces are

$$\frac{\partial D_y}{\partial y}.\delta y.\delta x.\delta z \text{ and } \frac{\partial D_z}{\partial z}.\delta z.\delta x.\delta y$$

$$\text{So}\quad \iint D.dS=\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)\delta x.\delta y.\delta z$$
$$=4\pi\rho.\delta x.\delta y.\delta z.$$

$$\text{Hence}\quad \frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}=4\pi\rho$$

$$\text{or}\quad div\ \mathbf{D}=4\pi\rho$$

In free space $\rho=0$, $div\ \mathbf{D}=\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}=0$ ... (i)

If B be the ***magnetic induction***, similar equation for a magnetic field will be

$$div\ \mathbf{B}=\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0 \qquad \text{... (ii)}$$

Maxwell's Displacement current : Maxwell considered that any change in electric induction in a medium constitutes an electric current arising out of displacement of charge. So he assumed that whenever there is a change of electric field in a medium there is the consequent production of an attendant magnetic field. There is also the reverse effect. Such a magnetic field is related to the charge in electric field in a similar way as the case of production of a magnetic field by a current flowing through a conductor. When an electron moves through a conductor, a magnetic field is produced. If an electric charge flows on to an insulated conductor, as in the case of a charge given to a condenser plate, it constitutes a pulse of electric current inside the dielectric. Such a current has been given the name of ***displacement current*** as distinguished from conduction current.

When a charge flows into a condenser by means of conducting leads, there is conduction current in the leads. According to Maxwell's conception this current is considered to be continuous through the dielectric and its magnitude is determined by the rate of change of electric displacement. This displacement current however ceases after a short time when the displacement (D) attains its maximum value and becomes constant.

A hydrostatic analogy may be thought of. Let a closed vessel filled with water and having an elastic partition inside be provided with two pipes on the opposite walls. If water be pushed through one of the pipes, the pressure on the partition will cause water to come out through the other pipe. As such there is a flow of water through the pipes, which however ceases when the displacement of the elastic partition reaches its elastic limit. In fact, while the displacement increases there is a current of water.

Maxwell's postulate regarding the conduction and displacement currents is that the algebraic sum of the two currents through any closed surface is zero. Let a charge Q be carried to a condenser plate. The conduction current entering the plate surface is measured as $i=\frac{dQ}{dt}$. Let the displacement current leaving the plate be i_D. According to Maxwell's postulate $i-i_D=0$, or $i=i_D$. If A be the area of the plate and I_D the density of displacement current, then $i_D=I_D.A$. Again $Q=\sigma A$, where σ is the surface density of charge on the plate. So

$$I_D=\frac{i_D}{A}=\frac{i}{A}=\frac{1}{A}\frac{\partial Q}{\partial t}=\frac{\partial \sigma}{\partial t}$$

Displacement D is related to σ by the equation $D=4\pi\sigma$, so

$$I_D=\frac{\partial \sigma}{\partial t}=\frac{1}{4\pi}.\frac{\partial D}{\partial t}=\frac{K}{4\pi}.\frac{\partial E}{\partial t}, \text{ since } D=KE.$$

This is the displacement current per unit area due to change in electric intensity in a medium.

EQUATIONS CONCERNING AMPERE'S THEOREM : The line integral of a magnetic field round any closed path is obtained from Ampere's circuital theorem as

$$\int_{o} H \cos\theta . \, dl = 4\pi I$$

Considering both the conduction and displacement currents, the above equation should be written as

$$\int_{o} H \cos\theta . \, dl = 4\pi I + 4\pi \int\int I_D . \, dS$$

$$= 4\pi I + \int\int \frac{\partial D}{\partial t} . \, dS.$$

If the conduction current is zero, as it is in a dielectric,

$$\int_{o} H \cos\theta . \, dl = \int\int \frac{\partial D}{\partial t} . \, dS = K \int\int \frac{\partial E}{\partial t} . \, dS.$$

For evalution of $\int_{o} H \cos\theta . \, dl$, consider a very small circuit *abcd* parallel to *y-z* plane (Fig. 10·2) and of sides of length δy and δz respectively.

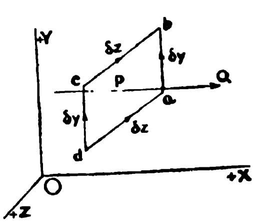


Fig. 10·2

Let H_y and H_z be the magnetic intensities along *ab* and *ad* respectively. The field along *dc* is $H_y + \frac{\partial H_y}{\partial z} . \delta z$ and that along *bc* is $H_z + \frac{\partial H_z}{\partial y} . \delta y$. The displacement D_x is along *PQ* normal to the plane of the circuit. If a unit pole is circulated around *abcd*, the work done on the pole will be

$$H_y \delta y + \left(H_z + \frac{\partial H_z}{\partial y} . \, \delta y \right) \delta z - \left(H_y + \frac{\partial H_y}{\partial z} . \, \delta z \right) \delta y - H_z . \delta z$$

$$\text{So work} = \left(\frac{\partial Hz}{\partial y} - \frac{\partial H_y}{\partial z} \right) . \delta y . \delta z.$$

This is equal to $\int\int \frac{\partial D_x}{\partial t} . dS = \frac{\partial D_x}{\partial t} . \delta y . \delta z.$

So we have, $\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{\partial D_x}{\partial t}$

Similarly if the rectangle *abcd* is taken in *xoz* and *xoy* planes respectively,

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{\partial D_y}{\partial t}$$

and, $$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{\partial D_z}{\partial t}. \qquad \text{(iii)}$$

In vector form, Curl $\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$, a general equation.

EQUATION FROM NEUMANN'S LAW : The electromotive force e round a circuit through which a magnetic flux is changing is given by $e = -\frac{\partial N}{\partial t}$.

Considering the induction through the area *abcd* (Fig. 10·2) to be along *PQ*, normal to the area, we have

$$N_x = B_x.\delta y.\delta z = \mu H_x.\delta y.\delta z$$

$$\text{So } emf \; -\frac{\partial N_x}{\partial t} = -\mu \frac{\partial H_x}{\partial t}.\delta y.\delta z.$$

Let E_y and E_z be the electric intensities along *ab* and *ad* respectively produced by change in the magnetic flux according to Maxwell's concept. The work done in circulating unit charge round the closed path *abcd* is a measure of *emf* and this is obtained as,

$$E_y.\delta y + \left(E_z + \frac{\partial E_z}{\partial y}\,\partial y\right)\delta z - \left(E_y + \frac{\partial E_y}{\partial z}.\delta z\right)\delta y - E_z.\delta z.$$

$$= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right).\delta y.\delta z$$

Equating this to $-\mu\frac{\partial H_x}{\partial t}.\delta y.\delta z$, we get

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu\frac{\partial H_x}{\partial t}$$

Similarly if the rectangle *abcd* is taken in planes parallel to *xoz*, *xoy* respectively,

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu\frac{\partial H_y}{\partial t} \qquad \text{(iv)}$$

and, $$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu\frac{\partial H_z}{\partial t}$$

In vector form the general equation is Curl $\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

It should be noted that these equations apply when E and H are measured in the same units. If E is measured in e.s. unit and H in e.m. unit the equation should be modified as

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\mu}{c}\frac{\partial H_x}{\partial t}$$, where c is the ratio of two units.

Similarly for the equations in (*iii*) the modification should be as

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = c.\frac{\partial D_x}{\partial t}.$$

The four sets of equations (*i*), (*ii*), (*iii*) and (*iv*) form *Maxwell's field equations.*

X-2. ELECTRO-MAGNETIC WAVE

Wave equation : Maxwell's field equations may be combined to show how a particular vector changes with respect to time and position.

Taking the first equation from set (*iii*)

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{\partial D_x}{\partial t} = K\frac{\partial E_x}{\partial t}.$$

On differentiating, $K\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial}{\partial y}\left(\frac{\partial H_z}{\partial t}\right) - \frac{\partial}{\partial z}\left(\frac{\partial H_y}{\partial t}\right)$

Substituting from (*iv*) the values of $\frac{\partial H_z}{\partial t}$ and $\frac{\partial H_y}{\partial t}$, we get

$$K\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial}{\partial y}\left[\frac{1}{\mu}\left(-\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y}\right)\right] - \frac{\partial}{\partial z}\left[\frac{1}{\mu}\left(-\frac{\partial E_x}{\partial z} + \frac{\partial E_z}{\partial x}\right)\right]$$

or $$K\frac{\partial^2 E_x}{\partial t^2} = \frac{1}{\mu}\left[-\frac{\partial^2 E_y}{\partial x.\partial y} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{\partial^2 E_z}{\partial x.\partial z}\right]$$

or $$K\mu\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{\partial}{\partial x}\left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right]$$

From equation (*i*) we know that the last portion of the right hand side of the above equation is zero, hence

$$\left.\begin{aligned}
&\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}=K\mu\frac{\partial^2 E_x}{\partial t^2}\\
\text{Similarly,}\quad &\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial y^2}+\frac{\partial^2 E_y}{\partial y^2}=K\mu.\frac{\partial^2 E_y}{\partial t^2}\\
\text{and,}\quad &\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+\frac{\partial^2 E_z}{\partial z^2}=K\mu.\frac{\partial^2 E_z}{\partial t^2}
\end{aligned}\right\} \quad \cdots\cdots \quad (v)$$

Writing in a single equation in vector form $\nabla^2\mathbf{E}=K\mu.\frac{\partial^2\mathbf{E}}{\partial t^2}$

Again from equations of set (*iv*) by similar treatment we get,

$$\left.\begin{aligned}
&\frac{\partial^2 H_x}{\partial x^2}+\frac{\partial^2 H_x}{\partial y^2}+\frac{\partial^2 H_x}{\partial z^2}=K\mu.\frac{\partial^2 H_x}{\partial t^2}\\
&\frac{\partial^2 H_y}{\partial x^2}+\frac{\partial^2 H_y}{\partial y^2}+\frac{\partial^2 H_y}{\partial z^2}=K\mu.\frac{\partial^2 H_y}{\partial t^2}\\
&\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}+\frac{\partial^2 H_z}{\partial z^2}=K\mu.\frac{\partial^2 H_y}{\partial t^2}
\end{aligned}\right\} \quad \cdots\cdots \quad (vi)$$

In a single equation in vector form, $\nabla^2\mathbf{H}=K\mu.\frac{\partial^2\mathbf{H}}{\partial t^2}$

The two sets of equations in (*v*) and (*vi*) form the general wave equations. Any of these satisfies the differential equation,

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}=\frac{1}{v^2}.\frac{\partial^2 P}{\partial t^2}$$

This equation indicates that P changes in a wave character and the velocity of wave motion of P is v, which is equal to $1/\sqrt{K\mu}$. Thus we may conclude that in general any variation of electro-magnetic vectors E and H causes their propagation with a velocity of $1/\sqrt{K\mu}$ through the medium.

This should be noted that the velocity of wave propagation given by the formula $v=1/\sqrt{K\mu}$ is applicable where K and μ are both measured in the same system of units. If K is expressed in *e.s.* unit and μ in *e.m.* unit, the velocity v is obtained as $v=c/\sqrt{K\mu}$, where c is the ratio of two units. For vacuum in the two units respectively $K_0=1$ and $\mu_0=1$, and hence v is numerically equal to and is same as c.

The wave equations obtained in the foregoing section may also be obtained by application of vectorial methods.

WAVE EQUATION BY VECTOR TREATMENT : Maxwell's field equations may be expressed as vector forms shown below.

Maxwell's field equations :

(i) From Gauss's theorem in electrostatics in free space

$$div\ \mathbf{D} = \nabla.\mathbf{D} = K.\nabla.\mathbf{E} = 0$$

(ii) From Gauss's theorem in magnetism

$$div\ \mathbf{B} = \nabla.\mathbf{B} = \mu.\nabla.\mathbf{H} = 0$$

(iii) From Ampere's theorem for Maxwell's displacement current expressed in circuital form

$$Curl\ \mathbf{H} = \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = K.\frac{\partial \mathbf{E}}{\partial t}$$

(iv) Applying Neumann's law to Maxwell's postulate regarding creation of electric field by change of magnetic flux

$$Curl\ \mathbf{E} = \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu\frac{\partial \mathbf{H}}{\partial t}$$

Wave equations may be deduced from these four equations, as shown below.

(A). From *(iii)* $\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \left(K\frac{\partial \mathbf{E}}{\partial t}\right) = K\frac{\partial}{dt}(\nabla \times \mathbf{E})$

Since from *(iv)* we get, $\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}$, hence

$$\nabla \times (\nabla \times \mathbf{H}) = K\frac{\partial}{\partial t}\left(-\mu\frac{\partial \mathbf{H}}{\partial t}\right) = -K\mu\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

Again considering that the left hand side of the above equation is the vector product of three vectors,

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla(\nabla.\mathbf{H}) - (\nabla.\nabla)\mathbf{H} = -\nabla^2\mathbf{H},$$

since $\nabla.\mathbf{H} = 0$

$$\text{Hence} \quad \nabla^2\mathbf{H} = K\mu\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

(B). Similarly from *(iv)*, $\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\mu\frac{\partial \mathbf{H}}{\partial t}\right)$

$$= -\mu\frac{\partial}{\partial t}(\nabla \times \mathbf{H})$$

Substituting from (*iii*) for $(\nabla \times \mathbf{H})$,

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial t}\left(K \frac{\partial \mathbf{E}}{\partial t}\right) = -K\mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Considering that the left hand side of the above equation is the vector product of three vectors,

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla . \mathbf{E}) - (\nabla . \nabla)\mathbf{E} = -\nabla^2 \mathbf{E}$$

Since $\nabla . \mathbf{E} = 0$

Hence $\nabla^2 \mathbf{E} = K\mu \dfrac{\partial^2 \mathbf{E}}{\partial t^2}$

$\nabla^2 \mathbf{H} = K\mu \dfrac{\partial^2 \mathbf{H}}{\partial t^2}$ and $\nabla^2 \mathbf{E} = K\mu \dfrac{\partial^2 \mathbf{E}}{\partial t^2}$ are the wave equations. These indicate that any variation of magnetic intensity and electric intensity vectors causes them to advance through a medium in a wave form with a velocity $v = 1/\sqrt{K\mu}$. If the values of K and μ are in two different systems of units respectively then the velocity is given by $v = c/\sqrt{K\mu}$, where c is the ratio of units. If the equation is applied to vacuum for which $K_0 = 1$ in *e.s.* unit and $\mu_0 = 1$ in *e.m.* unit, v becomes numerically equal to c.

Plane Electrc-magnetic waves : Let us consider a simple case in which E and H are functions of x and t only. So these quantities will be constant over any plane, say *yoz* plane, at any given time and the derivatives of E and H with respect to y and z will be zero. The equations in (v) reduces to

$$\frac{\partial^2 E_x}{\partial x^2} = K\mu \frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_y}{\partial x^2} = K\mu \frac{\partial^2 E_y}{\partial t^2}$$

$$\frac{\partial^2 E_z}{\partial x^2} = K\mu \frac{\partial^2 E_z}{\partial t^2}$$

The general solution of any of these equations is of the form

$$E_z = f_1(x - vt) + f_2(x + vt),$$

where f_1 and f_2 are arbitrary functions and $v = 1/\sqrt{K\mu}$.

$E_z = f_1(x - vt)$ denotes a wave motion in the positive direction of x and $E_z = f_2(x + vt)$ in a direction opposite to it.

There will be likewise (considering the equation in *vi*) the magnetic intensity wave denoted as

$$H_z = f_1(x - vt) + f_2(x + vt)$$

The equations relate to plane waves, *i.e.* the wave in which the intensity is the same over a plane at right angles to the direction of propagation at any instant.

TRANSVERSE NATURE OF WAVES: It has been assumed that waves generate in the y-z plane and travel in the x-direction. Intensity over the whole plane is the same at any instant. Hence the derivatives of E and H with respect to y and z are zero. Applying these in equation (*iv*), we get

$$\mu \frac{\partial H_x}{\partial t} = 0, \quad \text{so } H_x = 0$$

$$\mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} \quad \text{and} \quad \mu \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x}$$

Applying similar arguments, we get from equations (*iii*)

$$K \frac{\partial E_x}{\partial t} = 0, \quad \text{so} \quad E_x = 0$$

$$K \frac{\partial E_y}{\partial t} = -\frac{\partial H_x}{\partial x} \quad \text{and} \quad K \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x}$$

H_x and E_x are both zero, this means that the electric and magnetic intensity vectors both lie in the plane of the wave.

Now let us consider the case in which the electric intensity is parallel to the z-axis. This means that $E_y = 0$.

From the relation $K\dfrac{\partial E_y}{\partial t} = -\dfrac{\partial H_z}{\partial x}$, we should conclude that if $E_y = 0$, H_z vanishes and we are left with H_y only. We are therefore to conclude that if the electric intensity is parallel to the z-axis, the magnetic intensity in the wave is parallel to y-axis. So we find that electric and magnetic intensity vectors are mutually at right angles and both are at right angles to the directions of propagation. This means that the **waves are transverse.** Such waves in which the electric and magnetic

intensity vectors are confined to one plane are known as *plane-polarised.* The plane of polarisation is considered according to convention to be the plane of electric intensity.

Sinusoidal waves : Let us consider a plane wave in which E and H vary in a simple harmonic manner. The wave equations in such a case may be written as

$$E_z = E_0 \sin\frac{2\pi}{\lambda}(x - vt)$$

$$\text{and} \quad H_y = H_0 \sin\frac{2\pi}{\lambda}(x - vt)$$

Consider any one of the vectors, say E_z, we find that at the instant of reckoning time, *i.e.* when $t=0$,

$$E_z = E_0 \sin\frac{2\pi}{\lambda}x$$

This equation gives the value of E_z at all points in a plane parallel to xoz plane and it is same everywhere on the plane at any instant. It is represented by the ordinate of the curve shown (Fig. 10·3). E_0 is the maximum value of E_z. At any point along Ox, the magnitude of E_z at any particular instant is determined by the value of x. If x be increased by λ, the equation changes to

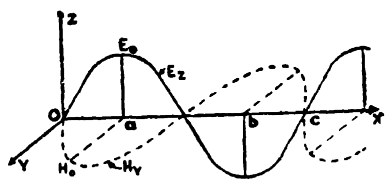


Fig. 10·3

$$E_z = E_0 \sin\frac{2\pi}{\lambda}(x+\lambda) = E_0 \sin\left(\frac{2\pi}{\lambda}x + 2\pi\right)$$

This shows that the curve repeats itself at a distance of λ. λ is called the *wave length,* represented in the curve by *Oc.* If the wave travels the distance λ in time t, $v=\lambda/T$. Substituting this in the equation for E_z, we get

$$E_z = E_0 \sin\frac{2\pi}{\lambda}\left(x - \frac{\lambda}{T}t\right) = E_0 \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

$$\text{Similarly,} \quad H_y = H_0 \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{\lambda}\right)$$

RELATION BETWEEN TWO INTENSITIES: It has been shown that for a wave in the *yoz* plane, $E_x=0$ and

$$\frac{\partial E_z}{\partial x}=\mu\frac{\partial H_y}{\partial t}$$

Differentiating the foregoing sinusoidal equation for E_z

$$\frac{\partial E_z}{\partial x}=\frac{2\pi}{\lambda}E_o\cos 2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)$$

Therefore $\dfrac{\partial H_y}{\partial t}=\dfrac{2\pi}{\mu\lambda}E_o\cos 2\pi\left(\dfrac{x}{\lambda}-\dfrac{t}{T}\right)$

Integrating, $H_y=-\dfrac{T}{\mu\lambda}E_o\sin 2\pi\left(\dfrac{x}{\lambda}-\dfrac{t}{T}\right)=-H_o\sin 2\lambda\left(\dfrac{x}{\lambda}-\dfrac{t}{T}\right)$

Hence the amplitude of H_y is $H_o=\dfrac{TE_o}{\mu\lambda}$

Since $\dfrac{T}{\lambda}=\dfrac{1}{v}=\sqrt{K\mu}$, so $H_o=\sqrt{K/\mu}\,E_o$

Hence we have the relation $\sqrt{\mu}\,H_o=\sqrt{K}E_o$,

when the quantities involved are measured in the same system of units.

If H_o is measured in *e.m.* units and E_o in *e.s.* units, for vacuum $\mu_0=K_o=1$ and $H_0=E_o$ in vacuum.

For a sinusoidal wave the r.m.s. values $\overline{H}$ and $\overline{E}$ are related to the maximum values as

$$\overline{H}=\frac{H_o}{\sqrt{2}},\ \overline{E}=\frac{E_o}{\sqrt{2}},\ \text{hence}\ \frac{\overline{H}}{\overline{E}}=\frac{H_o}{E_o}=\sqrt{\frac{K}{\mu}}.$$

X-3. POYNTING'S THEOREM

Energy of waves: Energy contained in unit volume in a space of dielectric constant K and electric intensity E_z is $\dfrac{KE_z{}^2}{8\pi}$. In the case of a plane wave of E_z varies harmonically at every point of the medium, the mean of $E_z^2=E_o^2/2$, where E_o is the amplitude. Hence

$$\text{Average}\ \frac{KE_z{}^2}{8\pi}=\frac{KE_0{}^2}{16\pi}.$$

Similarly in a field of magnetic intensity H_y, the mean energy per unit volume is $\frac{\mu H_0^2}{16\pi}$.

It has been shown that $\sqrt{\mu}H_0=\sqrt{K}E_0$, hence we get $\mu H_0^2=KE_0^2$, so the *energy per unit volume due to the electric and the magnetic components of the wave are equal.*

$$\text{Total energy density}=\frac{KE_0^2+\mu H_0^2}{16\pi}=\frac{KE_0^2}{8\pi}=\frac{\mu H_0^2}{8\pi}.$$

Poynting's vector : It relates the flow of energy per unit time through unit area of a plane perpendicular to the direction of propagation of wave. This is a measure of *intensity of radiation.*

Consider the plane wave in the *yoz* plane, propagating in *x*-direction. If *v* is the velocity of waves, the flow of energy in unit time over unit area will be all energy contained in a volume of unit cross section and length *v*. The amount of energy contained in this volume is

$$S=\left(\frac{KE_0^2}{16\pi}+\frac{\mu H_0^2}{16\pi}\right)v=\frac{KE_0^2}{8\pi}v=\frac{c\sqrt{K/\mu}}{8\pi}E_0^2 \qquad \ldots \qquad \text{(i)}$$

$$\text{since } v=\frac{c}{\sqrt{K\mu}}$$

$$\text{Similarly} \quad S=\frac{c\sqrt{\mu/K}}{8\pi}H_0^2 \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

From (*i*) and (*ii*)

$$S=\frac{c}{8\pi}.E_0H_0$$

This amount of energy which flows per unit time through unit area is the intensity of the wave. This is found to be directly proportional to the square of the electric or magnetic field strengths.

The intensity vector S is known as *Poynting's vector* and is represented along a straight line drawn perpendicular to

E and **H**. In a plane wave E and H are perpendicular to one another the magnitude of **S** given by

$$S=\frac{cE_0H_0}{8\pi}$$

In general, where the vectors are inclined at an angle θ, S is given by

$$S=\frac{cE_0H_0 \sin\theta}{8\pi}$$

In vector notation, $\mathbf{S}=\dfrac{c.\mathbf{E}_0\times\mathbf{H}_0}{8\pi}$

E_0 and H_0 are the amplitudes of respective intensities. If they are respectively expressed in *e.s* units and *e.m.* units, S is obtained in ergs/cm^2 per second.

According to *Poynting's theorem*, the direction of flow of energy is determined by the direction of electrical and magnetic intensities, its value being proportional to their product. If the direction of propagation of one of the quantities be reversed, the direction of flow of energy, *i.e.* of the wave propagation is reversed.

ILLUSTRATIVE EXAMPLE: *A kilo-watt lamp is radiating energy uniformly. Calculate the intensity of electric field at a distance of 5 metres from it.*

Solution: 1 KW = 1000 watts = 1000 joules/sec = 10^{10} ergs/sec.

At a distance of 5 metres, the spherical surface over which the energy is spread is equal to $4\pi\times(5\times100)^2$, *i.e.* $10^6\pi$ sq. cm.

Flux per sq. cm. is $=\dfrac{10^{10}}{10^6\pi}=\dfrac{10^4}{\pi}$

$$S=\frac{c\sqrt{K/\mu}}{8\pi}E_0^2$$

For air K = μ = 1, hence

$$\frac{c}{8\pi}E_0^2=\frac{10^4}{\pi}$$

$$\text{or} \quad E_0^2 = \frac{10^4 \times 8}{c} = \frac{10^4 \times 8}{3 \times 10^{10}}$$

$$\text{or} \quad E_0 = \frac{2\sqrt{2} \times 10^{-3}}{\sqrt{3}} \text{ e.s.u.}$$

$$\text{or} \quad \overline{E} = \frac{E_0}{\sqrt{2}} = \frac{2 \times 10^{-3}}{\sqrt{3}} \text{ e.s.u} = \frac{300 \times 2 \times 10^{-3}}{1{\cdot}73} \text{ volt/cm.}$$

or the average (R.M.S.) intensity is 0·345 volt/cm.

X-4. REFRACTIVE INDEX OF LIGHT

Relative index of refraction : According to wave theory, the relative refractive index of light when it passes from medium-1 to medium-2 is given by

$$ {}_1n_2 = \frac{\text{Velocity of light in medium-1}}{\text{Velocity of light in medium-2}}$$

$$\text{So} \quad {}_1n_2 = \frac{\text{Velocity of e.m. waves in medium-1}}{\text{Velocity of e.m. waves in medium-2}} = \frac{c/\sqrt{K_1\mu_1}}{c/\sqrt{K_2\mu_2}}$$

K_1, K_2 are in *e.s.* units and μ_1, μ_2 are in *e.m.* units. Since μ of a transparent medium is very nearly equal to 1, hence putting $\mu_1 = \mu_2 = 1$ in the above equation, we get

$$ {}_1n_2 = \sqrt{K_2/K_1}$$

If the light passes from vacuum to another medium whose permittivity is K, in such a case the absolute refractive index of the medium is given by $n = \sqrt{K}$ or $n^2 = K$.

This result is found to be in agreement with experimental results for gases. But there are wide variations for many cases for solids and liquids, specially distilled water. This is attributed to the absorption of waves, which modifies the refractive index considerably. Further, for distilled water its permittivity as measured at different frequencies with alternating current varies widely. These facts explain the variation of n^2 from K.

X-5. WAVES AT THE SEPERATION OF TWO MEDIA

Reflection and Refraction : Let a plane sinusoidal electromagnetic wave travelling along *PO* (Fig. 10·4) in a medium of permittivity of K_1 meet a second medium of permittivity K_2.

at O at an angle θ with the normal to the boundary. The incident beam is in the x-y plane.

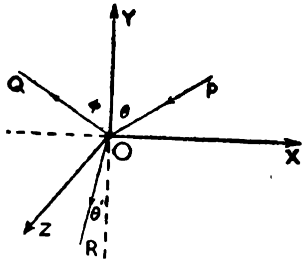


Fig. 10·4

The beam is reflected along OQ at an angle ϕ with the normal. The refracted beam is along OR at an angle θ' with the normal in the second medium. The boundary of the two media, *i.e.* the surface of separation is in the x-z plane and in this plane $y=0$.

Let the co-ordinates of points P in the incident beam be (x, y, z) and its direction-cosines with respect to the coordinate axes be l_1, m_1, n_1 respectively. As such the distance $OP=l_1x+m_1y+n_1z$.

Let us consider the wave-equation in the form

$$E=E_0 \sin 2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)$$

$$\text{or} \quad E=E_0 \sin\left(\frac{2\pi x}{\lambda}-\frac{2\pi}{T}t\right)$$

$$\text{or} \quad E=E_0 \sin\left(\frac{\omega x}{c}-\omega t\right)$$

$$\text{or} \quad E=E_0 \sin(\alpha x-\omega t)$$

$\omega=2\pi f$, f being the frequency and $\alpha=\omega/c$. $E_0 \sin(\alpha x-\omega t)$ may be considered as the imaginary part of the expression $E_0e^{j(\alpha x-\omega t)}$. We shall consider the z-component of the wave which is tangential to the surface of separation. Let the distance $OP=x'$ and let us assume that the z-component along OP may be represented as

$$E_1=E_0e^{j(\alpha_1x'-\omega t)}$$

$$\text{or} \quad E_1=E_0e^{j[\alpha_1(l_1x+m_1y+n_1z)-\omega_1t]}$$

Again consider that the direction-cosines of reflected and refracted beams be (l_2, m_2, n_2) and (l_3, m_3, n_3) respectively.

The z-components of these waves be represented in the same way as

$$E_2 = E'_0 e^{j[\alpha_2(l_2x + m_2y + n_2z) - \omega_2 t]}$$

$$\text{and} \quad E_3 = E''_0 e^{j[\alpha_3(l_3x + m_3y + n_3z) - \omega_3 t]}$$

At the boundary $y=0$. According to boundary conditions the tangential components are equal, so $E_1 + E_2 = E_3$. To satisfy this the coefficients of x, y, z and t must be equal in the three equations.

(*i*) Taking the coefficients of t to be equal, we get $\omega_1 = \omega_2 = \omega_3$, So $f_1 = f_2 = f_3$. This means that *the frequency remains unchanged by reflection and refraction.*

Since $\omega_1 = \omega_2$ and velocities of incident and reflected beams, which are in the same media, are same so $\omega_1/c = \omega_2/c$, *i.e.* $\alpha_1 = \alpha_2$.

(*ii*) Equating coefficients of z, we get $\alpha_1 n_1 = \alpha_2 n_2 = \alpha_3 n_3$. But the incident beam being perpendicular to the x-z plane, $n_1 = 0$, so $\alpha_1 n_1 = \alpha_2 n_2 = \alpha_3 n_3 = 0$. Hence $n_2 = n_3 = 0$. This means that all the beams are in the x-y plane which contains the normal. This is the *first law of reflection.*

(*iii*) Equating coefficients of x, we get $\alpha_1 l_1 = \alpha_2 l_2 = \alpha_3 l_3$.

$$\text{So} \quad \frac{l_1}{l_2} = \frac{\alpha_2}{\alpha_2}, \quad \text{since } \alpha_2 = \alpha_1 \quad \text{hence } l_2 = l_1$$

But $l_1 = \cos(90° - \theta) = \sin\theta$ and $l_2 = \cos(90° - \phi) = \sin\phi$

Hence $\sin\theta = \sin\phi$, *i.e.* $\theta = \phi$, angle of incidence is equal to the angle of reflection. This is *second law of reflection.*

(*iv*) Again since $\alpha_1 l_1 = \alpha_3 l_3$, so $\frac{\alpha_3}{\alpha_1} = \frac{l_1}{l_3}$.

But $l_1 = \cos(90° - \theta) = \sin\theta$ and $l_3 = \cos(90° - \theta') = \sin\theta'$

$$\text{So} \quad \frac{\alpha_3}{\alpha_1} = \frac{l_1}{l_3} = \frac{\sin\theta}{\sin\theta'}$$

$$\text{Considering that} \quad \frac{\alpha_3}{\alpha_1} = \frac{\omega_3}{c_3} \div \frac{\omega_1}{c_1} = \frac{c_1}{c_3}$$

We get $\dfrac{\sin\theta}{\sin\theta'} = \dfrac{c_1}{c_3} =$ Ratio of velocities in the two media.

This is *Snell's law of refraction.*

X-6. HERTZ'S EXPERIMENT

Detection of Electromagnetic radiation : Professor Hertz proved experimentally the existence of electro-magnetic radiation, *i.e* the propagation of electro-magnetic energy through space.

In his experimental arrangement Hertz took two square sheets of metal (40 cm. each way). These were placed in a vertically coplanar position with their centres about 60 cm. apart. These two formed an open-type capacitor. One highly polished brass sphere was attached to each of these sheets and their centres were kept separated by 2 to 3 cm. An induction coil was connected across these spheres so as to charge the capacitors. There were intermittent oscillatory discharges of the charge accumulated in the plates.

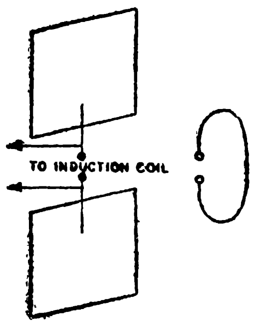


Fig. 10·5

In order to detect the radiation, Hertz used a circle of thick wire, almost closed having two sparking knobs at each end separated by a narrow gap. If this circular wire is placed in a vertical plane it will cut the lines of force associated with the waves propagating through it. An alternating *emf* will be induced round the circuit. When the potential difference of the knob is high, there is discharge in the form of small sparks. The detector will not respond if it set in a horizontal plane. Maximum response is obtained when the plane of the circle is perpendicular to the magnetic lines of force, *i.e.* at right angles to the plane of the capacitor plates.

If the natural period of the conducting circle which has its self-inductance and capacitance, is the same as the period of oscillations of the electro-magnetic waves determined by the frequency of oscillatory discharge, a maximum effect is produced due to resonance.

The explanation of the production of sparks may be found in the generation of induced *emf* in the circular wire when the

magnetic lines of force in the waves pass across the gap at right angles to it.

The electromagnetic waves produced by Hertz were of much lower frequency than those of light waves. But the properties of the electromagnetic waves deduced theoretically by Maxwell were actually verified by experiments. Hertz showed that electromagetic waves are reflected, refracted and they produce interference.

X-7. STANDING WAVES

Plane waves incident on a conductor : Consider a sinusoidal wave travelling in the negative direction of x. The wave is plane-polarised and the electric vector E_z is parallel to the y-z plane. The two fields comprising the wave are given by

$$E_z = E_0 \sin 2\pi\left(\frac{x}{\lambda}+\frac{t}{T}\right) \quad \cdots \quad \cdots \quad \text{(i)}$$

$$\text{and } H_y = H_0 \sin 2\pi\left(\frac{x}{\lambda}+\frac{t}{T}\right) \quad \cdots \quad \cdots \quad \text{(ii)}$$

Let the wave be incident normally on a conductor positioned at $x=0$, in the y-z plane. The equations of the waves on the plane are given by

$$E_z = E_0 \sin \frac{2\pi t}{T} \quad \text{and} \quad H_y = H_0 \sin \frac{2\pi t}{T}$$

Since an electric field cannot exist in a conductor, we may imagine that as a result of this incidence, an electric field is developed in the plane and this at all times is exactly equal and opposite to the field in the incident wave at $x=0$. This field may be attributed to the effect of the magnetic field on the plane giving rise to a reversed electric field. Thus there is a reflected wave travelling in opposite direction, *i.e.* in the positive direction of x.

The electric vector of the reflected wave is represented by

$$E'_z = E_0 \sin\left(\frac{x}{\lambda}+\frac{t}{T}\right) \quad \cdots \quad \text{(iii)}$$

This wave at the conductor ($x = 0$), becomes

$E_z = -E_0 \sin \frac{2\pi t}{T}$, the necessary value for the electric intensity at the conductor to be zero.

The resultant electric intensity is the sum of the two intensities. If it is denoted by E, then by adding (*i*) and (*ii*),

$$E = E_z + E'_z = E_0 \sin \frac{2\pi x}{\lambda}.\cos \frac{2\pi t}{T} = A \cos \frac{2\pi t}{T},$$

$$\text{where } A = 2E_0 \sin \frac{2\pi x}{\lambda}.$$

The electric field wave assumes a stationary character of the form $A \cos \frac{2\pi t}{T}$. The value of A is maximum ($2E_0$) at the points where x satisfies the condition $\sin \frac{2\pi x}{\lambda} = 1$. The values of x for such maximum values of E are obtained from the equation

$$\frac{2\pi x}{\lambda} = (2n+1)\frac{\pi}{2}, \text{ where } n = 0 \text{ or an integer.}$$

Therefore $x = (2n+1)\frac{\lambda}{4}$.

So maximum values of E occur at distances $\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots (2n+1)\frac{\lambda}{4}$ from the conductor. These are known as positions of ***antinodes***.

On the other hand, A becomes zero when $\sin \frac{2\pi x}{\lambda} = 0$. This will occur when $\frac{2\pi x}{\lambda} = 0$, or $n\pi$. This gives the relevant values of x as $0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2} \ldots \frac{n\lambda}{2}$. These are positions of *nodes*.

The electric field generated in the conductor gives rise to a displacement current and consequent magnetic field. This field, *i.e.* the reflected magnetic field is not reversed in phase but simply moves in opposite direction. This wave can be represented as

$$H_y = H_0 \sin 2\pi \left(-\frac{x}{\lambda} + \frac{t}{T}\right)$$

$$\text{or} \quad H'_y = -H_0 \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

The resultant of the two waves, incident and reflected waves, is obtained as

$$H = H_y + H'_y = 2H_0 \cos \frac{2\pi}{\lambda} x . \sin \frac{2\pi t}{T}$$

It may be noted that nodes and antinodes of the electric and magnetic vectors, determined respectively by sine and cosine functions, do not occur at coincident points. E has its antinode at the position where H has a node and vice versa.

It may be stated that since the magnetic field is not reversed, *i.e.* only one of the components is reversed in phase, according to Poynting's theorem the direction of propagation of energy is reversed, that is the wave travels away from the plane.

Lecher wire : A pair of parallel wires in connected to the terminals of capacitor in which an electric discharge is produced. The electromagnetic waves generated travel down the wires. The waves are reflected from the opposite ends of the wire and stationary waves are set up along the conductors. If the wires are open at the other end these terminals become potential nodes and current antinodes. The positions of the other potential antinodes along the length of the wires are found by using a neon lamp. The distance between the successive antinodal points will be $\lambda/2$. If f is the frequency of the oscillatory discharge, c is the velocity of propagation of waves and λ, the wave-length, then $c = f\lambda$. If λ

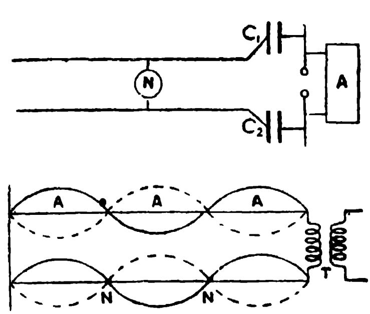


Fig. 10·6

is measured f can be determined, or conversely velocity of waves can be obtained from a knowledge of f.

Instead of an induction coil, a valve oscillator may be used as a source and the coupling between the source and the lines may be made by a suitable air-core transformer.

EXERCISES ON CHAPTER X

10·1. Write down the Maxwell's field equations and explain their physical interpretations. Derive expression for the velocity of electromagnetic wave propagation in an isotropic dielectric.

10·2. Show that the electromagnetic waves travel through a medium with a velocity $c=1/\sqrt{K\mu}$.

10·3. Derive an expression for the Poynting's vector and explain its significance. Explain Poynting's theorem.

10·4. Describe Hertz's experiment and hence show how the electromagnetic character of light can be established.

10·5. Obtain an expression showing the relation between the refractive index and dielectric constant of a medium.

10·6. Show that for a sinusoidal electromagnetic plane wave the electric and the magnetic intensity vectors are related as $\sqrt{\mu}H_o=\sqrt{K}E_o$.

10·7. Prove that the electromagnetic waves are transverse and the electric and magnetic intensity vectors are at right angles to one another and they are in the plane of the wave.

10·8. Obtain the laws of reflection and refraction of electro-magnetic waves.

10·9. Obtain the wave equations

$$\nabla^2\mathbf{H}=K\mu\frac{\delta^2\mathbf{H}}{\delta t^2} \text{ and } \nabla^2\mathbf{E}=K\mu\frac{\delta^2\mathbf{E}}{\delta t^2}$$

Prove that in a di-electric medium

$$\text{Curl } \mathbf{E}= -\frac{\delta \mathbf{B}}{\delta t}$$

INDEX

CORRIGENDA

Page/Line	*Correction*
45/1	Read $\sigma = i$
73/7	Read $\sin\frac{\theta}{2}$
77/4	Read $A^2H^2n^2/R$
77/8	Read $\sqrt{k^2-\omega^2}.\,t.$
83/10	Read $H' = N/a = K\theta/a$
162/5-6	Delete 'area......direction of'
173/3	Insert after line-3

$$\delta N = \left(\frac{2i}{x}-\frac{2i}{d-x}\right) l.\delta x = 2il\left[\frac{1}{x}-\frac{1}{d-x}\right]\delta x.$$

Total flux N through the whole space between the wires is given by

190/5	Read $e^{-\frac{R}{L}.t}$
228/14,17	Read $\frac{E_0}{R}\sin pt.$

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